On the Stability of the $n$-Dimensional Quadratic and Additive Functional Equation in Random Normed Spaces via Fixed Point Method

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Abstract

In this paper, we prove the stability in random normed spaces via fixed point method for the functional equation

$$f \left( \sum_{j=1}^{n} x_j \right) + (n - 2) \sum_{j=1}^{n} f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) = 0.$$  

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1 Introduction

S. M. Ulam [22] raised the stability problem of group homomorphisms and D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Hyers’ result was generalized by T. Aoki [1] for additive mappings and Th. M. Rassias [18] for linear mappings. Since then, a number of mathematicians have investigated the stability problems of functional equations extensively, see [2], [4], [5], [10]-[15].

Recall, almost all subsequent proofs in this very active area have used Hyers’ method, called a direct method. Namely, the function $F$, which is the solution of a functional equation, is explicitly constructed, starting from the given function $f$, by the formulae $F(x) = \lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right)$ or $F(x) = \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)$. In 2003, V. Radu [17] observed that the existence of the solution $F$ of a functional equation and the estimation of the difference with the given function $f$ can be obtained from the fixed point alternative. In 2008, D. Mihet and V. Radu [16] applied this method to prove the stability theorems of the Cauchy functional equation:

$$f(x + y) - f(x) - f(y) = 0$$

in random normed spaces. We call solutions of (1) by additive mappings. Now we consider the following $n$-dimensional quadratic and additive type functional equation

$$f \left( \sum_{j=1}^{n} x_j \right) + (n-2) \sum_{j=1}^{n} f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j) = 0.$$  

In 2006, K.-W. Jun and H.-M. Kim [9] investigated the stability of the functional equation (2) in classical normed spaces by using the direct method. Recently, the authors proved the stability of the functional equation (2) in fuzzy spaces [8] and used the fixed point method [7] to prove the stability for the functional equation (2) in Banach spaces. It is easy to see that the mappings $f(x) = ax^2 + bx$ is a solution of the functional equation (2). Every solution of the $n$-dimensional quadratic and additive functional equation is said to be a quadratic-additive mapping.

In this paper, using the fixed point method, we prove the stability for the functional equation (2) in random normed spaces.

2 Preliminaries

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [20,21]. Firstly, the space of all
probability distribution functions is denoted by

$$\Delta^+ := \{F : R \cup \{-\infty, \infty\} \to [0, 1] | F \text{ is left-continuous and nondecreasing on } R, \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1\}.$$ 

And let the subset $D^+ \subseteq \Delta^+$ be the set $D^+ := \{F' \in \Delta^+ | l^-F(+\infty) = 1\}$, where $l^-f(x)$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta^+$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in R$. The maximal element for $\Delta^+$ in this order is the distribution function $\varepsilon_0 : R \cup \{0\} \to [0, \infty)$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

**Definition 2.1** ([20]) A mapping $\tau : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangular norm (briefly, a continuous $t$-norm) if $\tau$ satisfies the following conditions:

(a) $\tau$ is commutative and associative;

(b) $\tau$ is continuous;

(c) $\tau(a, 1) = a$ for all $a \in [0, 1]$;

(d) $\tau(a, b) \leq \tau(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous $t$-norms are $\tau_P(a, b) = ab$, $\tau_M(a, b) = \min(a, b)$ and $\tau_L(a, b) = \max(a + b - 1, 0)$.

**Definition 2.2** ([21]) A random normed space (briefly, $RN$-space) is a triple $(X, \Lambda, \tau)$, where $X$ is a vector space, $\tau$ is a continuous $t$-norm, and $\Lambda$ is a mapping from $X$ into $D^+$ such that the following conditions hold:

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all $x$ in $X$, $\alpha \neq 0$ and all $t \geq 0$,

(RN3) $\Lambda_{x+y}(t+s) \geq \tau(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

If $(X, \| \cdot \|)$ is a normed space, we can define a mapping $\Lambda : X \to D^+$ by

$$\Lambda_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $(X, \Lambda, \tau_M)$ is a random normed space. This space is called the induced random normed space.

**Definition 2.3** Let $(X, \Lambda, \tau)$ be an $RN$-space.
(i) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if, for every \( t > 0 \) and \( \varepsilon > 0 \), there exists a positive integer \( N \) such that \( \Lambda_{x_n-x}(t) > 1 - \varepsilon \) whenever \( n \geq N \).

(ii) A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for every \( t > 0 \) and \( \varepsilon > 0 \), there exists a positive integer \( N \) such that \( \Lambda_{x_n-x_m}(t) > 1 - \varepsilon \) whenever \( n \geq m \geq N \).

(iii) An RN-space \( (X, \Lambda, \tau) \) is said to be complete if and only if every Cauchy sequence in \( X \) is convergent to a point in \( X \).

**Theorem 2.4** ([20]) If \( (X, \Lambda, \tau) \) is an RN-space and \( \{x_n\} \) is a sequence such that \( x_n \to x \), then \( \lim_{n \to \infty} \Lambda_{x_n}(t) = \Lambda_x(t) \).

3 The stability of the equation (2) for even \( n \)

We recall the fundamental result in the fixed point theory.

**Theorem 3.1** ([3] or [19]) Suppose that a complete generalized metric space \( (X, d) \), which means that the metric \( d \) may assume infinite values, and a strictly contractive mapping \( J : X \to X \) with the Lipschitz constant \( 0 < L < 1 \) are given. Then, for each given element \( x \in X \), either

\[
d(J^n x, J^{n+1} x) = +\infty, \forall n \in \mathbb{N} \cup \{0\}
\]

or there exists a nonnegative integer \( k \) such that:

1. \( d(J^n x, J^{n+1} x) < +\infty \) for all \( n \geq k \);
2. the sequence \( \{J^n x\} \) is convergent to a fixed point \( y^* \) of \( J \);
3. \( y^* \) is the unique fixed point of \( J \) in \( Y := \{y \in X, d(J^k x, y) < +\infty\} \);
4. \( d(y, y^*) \leq (1/(1-L))d(y, Jy) \) for all \( y \in Y \).

Throughout this paper, let \( X \) be a (real or complex) linear space and \( Y \) a Banach space. In this section, let \( n \) be an even number greater than 3. For a given mapping \( f : X \to Y \), we use the following abbreviations

\[
Df(x_1, x_2, \cdots, x_n) := \sum_{j=1}^{n} f \left( \sum_{j=1}^{n} x_j \right) + (n-2) \sum_{j=1}^{n} f(x_j) - \sum_{1 \leq i < j \leq n} f(x_i + x_j),
\]

\[
\hat{x} := (\sum_{i=1}^{n-1} x_i, -x_n)
\]

for all \( x, x_1, x_2, \cdots, x_n \in X \).

Now we will establish the stability for the functional equation (2) in random normed spaces via fixed point method for even \( n \).
Theorem 3.2 Let $X$ be a real linear space, $(Z, \Lambda, \tau_M)$ be an RN-space, $(Y, \Lambda, \tau_D)$ be a complete RN-space, $n$ be an even number greater than 3, and $\varphi : (X \setminus \{0\})^n \to Z$. Suppose that $\varphi$ satisfies one of the following conditions:

(i) \[ \Lambda_{\alpha \varphi(x_1, x_2, \ldots, x_n)}(t) \leq \Lambda_{\alpha \varphi(x_2, x_2, \ldots, x_n)}(t) \text{ for some } 0 < \alpha < 2, \]

(ii) \[ \Lambda_{\varphi(2x_1, x_2, \ldots, x_n)}(t) \leq \Lambda_{\varphi(x_1, x_2, \ldots, x_n)}(t) \text{ for some } 4 < \alpha \]

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$. If $f : X \to Y$ is a mapping such that

\[ \Lambda_{DF(x_1, x_2, \ldots, x_n)}(t) \geq \Lambda_{\varphi(x_1, x_2, \ldots, x_n)}(t) \tag{3} \]

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and $t > 0$ with $f(0) = 0$, then there exists a unique mapping $F : X \to Y$ such that

\[ DF(x_1, x_2, \ldots, x_n) = 0 \]

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

\[ \Lambda_{f(x) - F(x)}(t) \geq \begin{cases} M(x, (n - 2)(2 - \alpha)t) & \text{if } \varphi \text{ satisfies (i)}, \\ M(x, (n - 2)(\alpha - 4)t) & \text{if } \varphi \text{ satisfies (ii)} \end{cases} \tag{4} \]

for all $x \in X \setminus \{0\}$ and $t > 0$, where

\[ M(x, t) := \tau_M \{ \Lambda'_{\varphi(x)}(t), \Lambda'_{\varphi(-x)}(t) \}. \]

Moreover if $\alpha < 1$ and $\Lambda_{\varphi(x_1, x_2, \ldots, x_n)}$ is continuous in $x_1, x_2, \ldots, x_n$ under the condition (i), then $f \equiv F$.

**Proof.** We will prove the theorem in two cases, $\varphi$ satisfies the condition (i) or (ii).

**Case 1.** Assume that $\varphi$ satisfies the condition (i). Let $S$ be the set of all functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on $S$ by

\[ d(g, h) = \inf \left\{ u \in R^+ | \Lambda_{g(x) - h(x)}(ut) \geq M(x, t) \text{ for all } x \in X \setminus \{0\} \right\}. \]

Consider the mapping $J : S \to S$ defined by

\[ Jf(x) := \frac{f(2x) - f(-2x)}{4} + \frac{f(2x) + f(-2x)}{8} \]

then we have

\[ J^m f(x) = \frac{1}{2} \left( 4^{-m} \left( f(2^m x) + f(-2^m x) \right) + 2^{-m} \left( f(2^m x) - f(-2^m x) \right) \right) \]
for all \( x \in X \). Let \( f, g \in S \) and let \( u \in [0, \infty] \) be an arbitrary constant with \( d(g, f) \leq u \). From the definition of \( d \), (RN2), and (RN3), for the given \( 0 < \alpha < 2 \) we have

\[
\Lambda_{Jg(x) - Jf(x)} \left( \frac{\alpha u}{2} t \right) = \Lambda_{\frac{3(g(2x) - f(2x)) - g(-2x) - f(-2x)}{8}} \left( \frac{\alpha u}{2} t \right) \\
\geq \tau_M \left\{ \Lambda_{\frac{3(g(2x) - f(2x)) - g(-2x) - f(-2x)}{8}} \left( \frac{3\alpha ut}{8} \right), \Lambda_{\frac{g(-2x) - f(-2x)}{8}} \left( \frac{\alpha ut}{8} \right) \right\} \\
\geq \tau_M \left\{ \Lambda_{g(2x) - f(2x)}(\alpha ut), \Lambda_{g(-2x) - f(-2x)}(\alpha ut) \right\} \\
\geq \tau_M \left\{ \Lambda'_{\psi(2x)}(\alpha t), \Lambda'_{\psi(-2x)}(\alpha t) \right\} \\
\geq M(x, t)
\]

for all \( x \in X \setminus \{0\} \), which implies that

\[
d(Jf, Jg) \leq \frac{\alpha}{2} d(f, g).
\]

That is, \( J \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( 0 < \frac{\alpha}{2} < 1 \). By (3), we see that

\[
\Lambda_{f(x) - f(x)} \left( \frac{t}{2(n-2)} \right) = \Lambda_{\frac{(n+2)Df(x) - (n-2)Df(-x)}{4n(n-2)}} \left( \frac{t}{2(n-2)} \right) \\
\geq \tau_M \left\{ \Lambda_{\frac{(n+2)Df(x) - (n-2)Df(-x)}{4n(n-2)}} \left( \frac{(n+2)t}{4n(n-2)} \right), \Lambda_{\frac{Df(-x)}{4n}} \left( \frac{t}{4n} \right) \right\} \\
\geq \tau_M \left\{ \Lambda_{Df(x)}(t), \Lambda_{Df(-x)}(t) \right\} \\
\geq \tau_M \left\{ \Lambda'_{\phi(2x)}(t), \Lambda'_{\phi(-2x)}(t) \right\}
\]

for all \( x \in X \setminus \{0\} \). It means that \( d(f, Jf) \leq \frac{1}{2(n-2)} < \infty \) by the definition of \( d \). Therefore according to Theorem 3.1, the sequence \( \{J^m f\} \) converges to the unique fixed point \( F : X \to Y \) of \( J \) in the set \( T = \{g \in S \mid d(f, g) < \infty\} \), which is represented by

\[
F(x) := \lim_{m \to \infty} \left( \frac{f(2^m x + f(-2^m x)}{2^m} + \frac{f(2^m x) - f(-2^m x)}{2^{m+1}} \right)
\]

for all \( x \in X \). Since

\[
d(f, F) \leq \frac{1}{1 - \frac{\alpha}{2}} d(f, Jf) \leq \frac{1}{(n-2)(2-\alpha)}
\]

the inequality (4) holds. By (RN3), we have

\[
\Lambda_{DF(x_1, x_2, \ldots, x_n)}(t) \geq \tau_M \left\{ \Lambda_{Dg(x_1, x_2, \ldots, x_n)} \left( \frac{t}{4} \right), \Lambda_{(F-J)f(x_i)} \left( \frac{t}{4} \right), \right\} \\
\tau_M \left\{ \Lambda_{g(x_i, x_j)} \left( \frac{t}{4n} \right) : i = 1, \ldots, n \right\}, \\
\tau_M \left\{ \Lambda_{g(x_i, x_j)} \left( \frac{t}{2n(n-1)} \right) : 1 \leq i < j \leq n \right\}\right\} (5)
\]
for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) and \( m \in N \). The last three terms on the right hand side of the above inequality tend to 1 as \( m \to \infty \) by the definition of \( F \). Now consider that

\[
\Lambda_{DF}(x_1, x_2, \ldots; x_n)(t) \geq \tau_M \left\{ \Lambda_{DF}(\frac{4m}{16} t), \Lambda_{DF}(\frac{4m}{8} t), \Lambda_{DF}(\frac{2m}{8} t) \right\},
\]

which tends to 1 as \( m \to \infty \) by (RN3) and \( \frac{2}{\alpha} > 1 \) for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \). Therefore it follows from (5) that

\[
\Lambda_{DF}(x_1, x_2, \ldots; x_n)(t) = 1
\]

for each \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) and \( t > 0 \). By (RN1), this means that

\[
DF(x_1, x_2, \ldots, x_n) = 0
\]

for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \). Assume that \( \alpha < 1 \) and \( \Lambda'_{\varphi(x_1, x_2, \ldots; x_n)} \) is continuous in \( x_1, x_2, \ldots, x_n \). If \( N, a_1, b_1, a_2, b_2, \ldots, a_n, \text{ and } b_n \) are any fixed integers with \( a_1, a_2, \ldots, a_n \neq 0 \), then we have

\[
\lim_{m \to \infty} \Lambda'_{\varphi((2^m a_1 + b_1)x_1, (2^m a_2 + b_2)x_2, \ldots; (2^m a_n + b_n)x_n)}(t)
\]

\[
\geq \lim_{m \to \infty} \Lambda'_{\varphi\left((a_1 + \frac{b_1}{2^m})x_1, (a_2 + \frac{b_2}{2^m})x_2, \ldots; (a_n + \frac{b_n}{2^m})x_n\right)}\left(\frac{t}{\alpha m}\right)
\]

\[
\geq \lim_{N \to \infty} \Lambda'_{\varphi\left((a_1 + \frac{b_1}{2^N})x_1, (a_2 + \frac{b_2}{2^N})x_2, \ldots; (a_n + \frac{b_n}{2^N})x_n\right)}(Nt)
\]

\[
= \Lambda'_{\varphi(a_1 x_1, a_2 x_2, \ldots; a_n x_n)}(Nt)
\]

for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) and \( t > 0 \). Since \( N \) is arbitrary, we have

\[
\lim_{m \to \infty} \Lambda'_{\varphi((2^m a_1 + b_1)x_1, (2^m a_2 + b_2)x_2, \ldots; (2^m a_n + b_n)x_n)}(t)
\]

\[
\geq \lim_{N \to \infty} \Lambda'_{\varphi(a_1 x_1, a_2 x_2, \ldots; a_n x_n)}(Nt) = 1
\]

by (RN3) for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) and \( t > 0 \). From these, we get

\[
\Lambda_{n-1}(F(x) - f(x)) \left(\frac{(3n^2 - 7n + 6)t}{2}\right)
\]
\[
\geq \lim_{m \to \infty} \tau_M \left\{ \Lambda_{(F-f)((2^m+1)x, -2^m x, \ldots, -2^m x)}(t), \\
\Lambda_{(F-f)((2-n2^m+1)x)}(t), \Lambda_{(n-2)(F-f)((2^n+1)x)}((n-2)t), \\
\Lambda_{(n-1)(n-2)(F-f)(-2^m x)}((n-1)(n-2)t), \\
\Lambda_{(n-1)(n-2)(F-f)(-2^m)} \left( \frac{(n-1)(n-2)t}{2} \right) \right\}
\]
\[
\geq \lim_{m \to \infty} \tau_M \left\{ \Lambda'_{\varphi((2^m+1)x, -2^m x, \ldots, -2^m x)}(t), M(((2-n)2^m+1)x, (n-2)(2-\alpha)t), \\
M((2^m+1)x, (n-2)(2-\alpha)t), M(-2^m x, (n-2)(2-\alpha)t), \\
M(-2^m x, (n-2)(2-\alpha)t) \right\}
\]

\[= 1\]

for all \(x \in X \setminus \{0\}\). By (RN1) and (RN2), this means that \(f(x) = F(x)\) for all \(x \in X \setminus \{0\}\). Together with the fact \(f(0) = 0 = F(0)\), we obtain \(f \equiv F\).

**Case 2.** Assume that \(\varphi\) satisfies the condition (ii). Let the set \((S, d)\) be as in the proof of Case 1. Now we consider the mapping \(J : S \to S\) defined by

\[Jg(x) := g \left( \frac{x}{2} \right) - g \left( -\frac{x}{2} \right) + 2 \left( g \left( \frac{x}{2} \right) + g \left( -\frac{x}{2} \right) \right)\]

for all \(g \in S\) and \(x \in V\). Notice that

\[J^m g(x) = 2^{m-1} \left( g \left( \frac{x}{2^m} \right) - g \left( -\frac{x}{2^m} \right) \right) + \frac{4^m}{2} \left( g \left( \frac{x}{2^m} \right) + g \left( -\frac{x}{2^m} \right) \right)\]

and \(J^0 g(x) = g(x)\) for all \(x \in X\). Let \(f, g \in S\) and let \(u \in [0, \infty)\) be an arbitrary constant with \(d(g, f) \leq u\). From the definition of \(d\), (RN2), and (RN3), we have

\[\Lambda_{Jg(x)-Jf(x)} \left( \frac{4u \alpha}{t} \right) = \Lambda_{\varphi^{-1}(\varphi(x) - \varphi(f))} \left( \frac{3u \alpha}{t} \right), \Lambda_{\varphi^{-1}(\varphi(f) - \varphi(x))} \left( \frac{u \alpha}{t} \right)\]

\[\geq \tau_M \left\{ \Lambda_{\varphi^{-1}(\varphi(x) - \varphi(f))} \left( \frac{3u \alpha}{t} \right), \Lambda_{\varphi^{-1}(\varphi(f) - \varphi(x))} \left( \frac{u \alpha}{t} \right) \right\}\]

\[\geq \tau_M \left\{ \Lambda' \left( \frac{t}{\alpha} \right), \Lambda' \left( \frac{t}{\alpha} \right) \right\}\]

\[\geq M(x, t)\]

for all \(x \in X \setminus \{0\}\), which implies that

\[d(Jf, Jg) \leq \frac{4}{\alpha} d(f, g)\].
That is, $J$ is a strictly contractive self-mapping of $S$ with the Lipschitz constant $0 < \frac{4}{\alpha} < 1$. Moreover, by (3), we see that

$$\Lambda_{f(x)-Jf(x)} \left( \frac{t}{(n-2) \alpha} \right) = \Lambda - \frac{n+4}{2n(n-2)} Df(\frac{x}{2^n}) + \frac{n-4}{2n(n-2)} Df(\frac{x}{2^n}) \left( \frac{t}{(n-2) \alpha} \right)$$

$$\geq \tau_M \left\{ \Lambda - \frac{n+4}{2n(n-2)} Df(\frac{x}{2^n}) \left( \frac{(n+4)t}{2n(n-2) \alpha} \right), \right.\left. \Lambda - \frac{n-4}{2n(n-2)} Df(\frac{x}{2^n}) \left( \frac{(n-4)t}{2n(n-2) \alpha} \right) \right\}$$

$$\geq \tau_M \left\{ \Lambda' \varphi(\frac{x}{2^n}) \left( \frac{t}{\alpha} \right), \Lambda' \varphi(-\frac{x}{2^n}) \left( \frac{t}{\alpha} \right) \right\}$$

$$\geq M(x, t)$$

for all $x \in X\setminus\{0\}$. It means that $d(f, Jf) \leq \frac{1}{(n-2) \alpha} < \infty$ by the definition of $d$. Therefore according to Theorem 3.1, the sequence \{\text{J}^m f\} converges to the unique fixed point $F : X \to Y$ of $J$ in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by

$$F(x) := \lim_{m \to \infty} \left( 2^{m-1} \left( f \left( \frac{x}{2^m} \right) - f \left( -\frac{x}{2^m} \right) \right) + \frac{4^m}{2^m} \left( f \left( \frac{x}{2^m} \right) + f \left( -\frac{x}{2^m} \right) \right) \right)$$

for all $x \in X$. Since

$$d(f, F) \leq \frac{1}{1 - \frac{4}{\alpha}} d(f, Jf) \leq \frac{1}{(n-2)(\alpha - 4)}$$

the inequality (4) holds. Next we have the inequality (5) for all $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$ and $n \in N$. The last three terms on the right hand side of the inequality (5) tend to 1 as $m \to \infty$ by the definition of $F$. Now consider that

$$\Lambda_{DF(x_1, x_2, \ldots, x_n)} \left( \frac{t}{4} \right) \geq \tau_M \left\{ \Lambda_{2^{2m-1} Df(\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m})} \left( \frac{t}{16} \right), \Lambda_{2^{m-1} Df(\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m})} \left( \frac{t}{16} \right) \right\},$$

$$\Lambda_{2^{m-1} Df(\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m})} \left( \frac{t}{16} \right), \Lambda_{2^{m-1} Df(\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m})} \left( \frac{t}{16} \right) \right\}$$

$$\geq \tau_M \left\{ \Lambda' \varphi(x_1, x_2, \ldots, x_n) \left( \frac{\alpha^m t}{2^{m+3}} \right), \Lambda' \varphi(-x_1, -x_2, \ldots, -x_n) \left( \frac{\alpha^m t}{2^{m+3}} \right), \right.\left. \Lambda' \varphi(x_1, x_2, \ldots, x_n) \left( \frac{\alpha^m t}{2^{m+3}} \right), \Lambda' \varphi(-x_1, -x_2, \ldots, -x_n) \left( \frac{\alpha^m t}{2^{m+3}} \right) \right\}$$

which tends to 1 as $m \to \infty$ by (RN3) for all $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$. Therefore it follows from (5) that

$$\Lambda_{DF(x_1, x_2, \ldots, x_n)}(t) = 1$$

for each $x_1, x_2, \ldots, x_n \in X\setminus\{0\}$ and $t > 0$. By (RN1), this means that

$$DF(x_1, x_2, \ldots, x_n) = 0$$
for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \). It completes the proof of Theorem 3.2.

By the similar method used in the proof of Theorem 3.2, we get the following theorem.

**Theorem 3.3** Let \( X \) be a linear space, \( (Z, \Lambda', \tau_M) \) be an RN-space, \( (Y, \Lambda, \tau_M) \) be a complete RN-space, \( n \) be an even number greater than 3, and \( \varphi : X^n \to Z \). Assume that \( \varphi \) satisfies one of the following conditions:

(i) \( \Lambda'_{\varphi(x_1, x_2, \ldots, x_n)}(t) \leq \Lambda'_{\varphi(2x_1, 2x_2, \ldots, 2x_n)}(t) \) for some \( 0 < \alpha < 2 \),

(ii) \( \Lambda_{\varphi(2x_1, 2x_2, \ldots, 2x_n)}(t) \leq \Lambda_{\varphi(x_1, x_2, \ldots, x_n)}(t) \) for some \( 4 < \alpha \)

for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). If \( f : X \to Y \) is a mapping satisfying (3) for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \) with \( f(0) = 0 \), then there exists a unique quadratic-additive mapping \( F : X \to Y \) satisfying (4) for all \( x \in X \) and \( t > 0 \). Moreover if \( \alpha < 1 \) and \( \Lambda'_{\varphi(x_1, x_2, \ldots, x_n)} \) is continuous in \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) under the condition (i), then \( f \) is a quadratic-additive mapping.

Now we have the general Hyers-Ulam stability of the quadratic-additive functional equation (2) in the framework of normed spaces. If \( X \) is a normed space, then \( (X, \Lambda, \tau_M) \) is an induced random normed space. It leads us to get the following result.

**Corollary 3.4** Let \( X \) be a linear space, \( n \) be an even number greater than 3, \( Y \) be a complete normed-space, and \( \varphi : (X \setminus \{0\})^n \to [0, \infty) \). Suppose that \( \varphi \) satisfies one of the following conditions:

(i) \( \alpha \varphi(x_1, x_2, \ldots, x_n) \geq \varphi(2x_1, 2y_1, 2z_2, 2w) \) for some \( 0 < \alpha < 2 \),

(ii) \( \varphi(2x_1, 2x_2, \ldots, 2x_n) \geq \alpha \varphi(x_1, x_2, \ldots, x_n) \) for some \( 4 < \alpha \)

for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \). If \( f : X \to Y \) is a mapping such that

\[
\|Df(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n)
\]

for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) with \( f(0) = 0 \), then there exists a unique mapping \( F : X \to Y \) such that

\[
DF(x_1, x_2, \ldots, x_n) = 0
\]

for all \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) and

\[
\|f(x) - F(x)\| \leq \begin{cases} 
\frac{\Phi(x)}{(n-2)(1-\alpha)} & \text{if } \varphi \text{ satisfies (i)}, \\
\frac{\Phi(x)}{(n-2)(\alpha-4)} & \text{if } \varphi \text{ satisfies (ii)}
\end{cases}
\]

for all \( x \in X \setminus \{0\} \), where \( \Phi(x) \) is defined by

\[
\Phi(x) = \max(\varphi(\hat{x}), \varphi(-\hat{x})).
\]

Moreover, if \( 0 < \alpha < 1 \) under the condition (i), then \( f \equiv F \).
Now we have Hyers-Ulam-Rassias stability results of the quadratic-additive functional equation (2) in the framework of normed spaces.

**Corollary 3.5** Let $X$ be a normed space, $n$ be an even number greater than 3, $p \in R \setminus [1, 2]$ and $Y$ a complete normed space. If $f : X \to Y$ is a mapping such that
\[
\|Df(x_1, x_2, \cdots, x_n)\| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p
\]
for all $x_1, x_2, \cdots, x_n \in X \setminus \{0\}$ with $f(0) = 0$, then there exists a unique mapping $F : X \to Y$ such that
\[
DF(x_1, x_2, \cdots, x_n) = 0
\]
for all $x_1, x_2, \cdots, x_n \in X \setminus \{0\}$ and
\[
\|f(x) - F(x)\| \leq \begin{cases} 
  0 & \text{if } p < 0, \\
  \frac{n \|x\|^p}{(n-2)(2^{2p})} & \text{if } 0 \leq p < 1, \\
  \frac{n \|x\|^p}{(n-2)(2^p - 4)} & \text{if } p > 2
\end{cases}
\]
for all $x \in X \setminus \{0\}$.

**Proof.** If we denote by $\varphi(x_1, x_2, \cdots, x_n) = \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 3.2 with $\alpha = 2^p$.

**Corollary 3.6** Let $X$ be a normed space, $n$ be an even number greater than 3, $p \in R \setminus [1, 2]$ and $Y$ a complete normed space. If $f : X \to Y$ is a mapping such that
\[
\|Df(x_1, x_2, \cdots, x_n)\| \leq \sum_{i=1, x_i \neq 0}^n \|x_i\|^p
\]
for all $x_1, x_2, \cdots, x_n \in X$ with $f(0) = 0$, then there exists a unique quadratic-additive mapping $F : X \to Y$ satisfying (6) for all $x \in X \setminus \{0\}$.

**Proof.** If we denote by $\varphi(x_1, x_2, \cdots, x_n) = \sum_{i=1, x_i \neq 0}^n \|x_i\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 3.3 with $\alpha = 2^p$.

## 4 The stability of the equation (2) for odd $n$

In this section, let $n$ be an odd number greater than 3. We will establish the stability for the functional equations (2) in random normed spaces via fixed point method for odd $n$. 
Theorem 4.1 Let $X$ be a linear space, $(Z, \Lambda', \tau_M)$ be an RN-space, $(Y, \Lambda, \tau_M)$ be a complete RN-space, $n$ be an odd number greater than 3, and $\varphi : (X \setminus \{0\})^n \to Z$. Assume that $\varphi$ satisfies one of the following conditions (i) and (ii) in Theorem 3.3. If $f : X \to Y$ be a mapping satisfying (3) for all $x_1, x_2, \cdots, x_n \in X \setminus \{0\}$ and $t > 0$ with $f(0) = 0$, then there exists a unique mapping $F : X \to Y$ such that

$$DF(x_1, x_2, \cdots, x_n) = 0$$

for all $x_1, x_2, \cdots, x_n \in X \setminus \{0\}$ and

$$\Lambda_{f(x)-F(x)}(t) \geq \begin{cases} M'(x, \frac{(n-1)(2-\alpha)t}{2}) & \text{if } \varphi \text{ satisfies (i)}, \\ M'(x, \frac{(n-1)(\alpha-4)t}{2}) & \text{if } \varphi \text{ satisfies (ii)} \end{cases}$$

(7)

for all $x \in X$ and $t > 0$, where $\bar{x} := (\bar{x}, \cdots, \bar{x}, -x, \cdots, -x)$ and

$$M'(x, t) := \tau_M \{ \Lambda_{\varphi(x)}'(t), \Lambda_{\varphi(-x)}'(t) \}.$$ 

Moreover, if $\alpha < 1$ and $\Lambda_{\varphi(x_1, x_2, \cdots, x_n)}$ is continuous in $x_1, x_2, \cdots, x_n$ under the condition (i), then $f \equiv F$.

Proof. We will prove the theorem in two cases, $\varphi$ satisfies the condition (i) or (ii).

Case 1. Assume that $\varphi$ satisfies the condition (i). Let $S, J$ be as in Case 1 of the proof of Theorem 3.3. Now, we introduce a generalized metric on $S$ by

$$d(g, h) = \inf \left\{ u \in R^+ | \Lambda_{g(x)-h(x)}(ut) \geq M'(x, t) \text{ for all } x \in X \setminus \{0\} \right\}.$$ 

Notice that

$$\Lambda_{f(x)-Jf(x)} \left( \frac{t}{n-1} \right) = \Lambda_{\frac{Df(\overline{x})+Df(-\overline{x})}{2(n-1)^2}} \left( \frac{t}{n-1} \right) \geq \tau_M \left\{ \Lambda_{\frac{Df(\overline{x})}{2(n-1)^2}} \left( \frac{nt}{2(n-1)^2} \right), \Lambda_{\frac{Df(-\overline{x})}{2(n-1)^2}} \left( \frac{(n-2)t}{2(n-1)^2} \right) \right\} \geq \tau_M \{ \Lambda_{\varphi(\overline{x})}'(t), \Lambda_{\varphi(-\overline{x})}'(t) \} \geq \tau_M \{ \Lambda_{\varphi(x)}'(t), \Lambda_{\varphi(-x)}'(t) \}$$

for all $x \in X \setminus \{0\}$. It means that $d(f, Jf) \leq \frac{1}{n-1} < \infty$ by the definition of $d$. Using the similar method of Case 1 in the proof of Theorem 3.3, we can easily obtain the results of this case.
Case 2. Assume that \( \varphi \) satisfies the condition (ii). Let the set \((S,d)\) be as in the proof of Case 1 and let \( J \) be as in Case 2 of the proof of Theorem 3.3. Notice that

\[
\Lambda_f(x) - Jf(x) \left( \frac{2t}{(n-1)\alpha} \right)
\]

\[
= \Lambda \left( \frac{2Df(\tilde{\varphi}) - 2Df(\tilde{\varphi}) - Df(\tilde{\varphi}) + Df(\tilde{\varphi})}{(n-1)^2} \right) \left( \frac{2t}{(n-1)\alpha} \right)
\]

\[
\geq \tau_M \left\{ \Lambda_{Df(\tilde{\varphi})} \left( \frac{t}{\alpha} \right), \Lambda_{Df(\tilde{\alpha})} \left( \frac{t}{\alpha} \right) \right\}
\]

\[
\geq \tau_M \left\{ \Lambda'_{\varphi(\tilde{\varphi})} \left( \frac{t}{\alpha} \right), \Lambda'_{\varphi(\tilde{\alpha})} \left( \frac{t}{\alpha} \right) \right\}
\]

\[
\geq M'(x,t)
\]

for all \( x \in X \setminus \{0\} \). It means that \( d(f,Jf) \leq \frac{2}{(n-1)\alpha} < \infty \) by the definition of \( d \). Using the similar method of Case 2 in the proof of Theorem 3.3, we can easily obtain the result.

By the similar method used in the proof of Theorem 4.1, we get the following theorem.

**Theorem 4.2** Let \( X \) be a linear space, \((Z,N',\tau_M)\) be an RN-space, \((Y,\Lambda,\tau_M)\) be a complete RN-space, \( n \) be an odd number greater than 3, and \( \varphi : X^n \to Z \). Assume that \( \varphi \) satisfies one of the following conditions:

(i) \( N'_{\varphi(x_1,x_2,\ldots,x_n)}(t) \leq N'_{\varphi(2x_1,2x_2,\ldots,2x_n)}(t) \) for some \( 0 < \alpha < 2 \),

(ii) \( N'_{\varphi(2x_1,2x_2,\ldots,2x_n)}(t) \leq N'_{\varphi(x_1,\ldots,x_n)}(t) \) for some \( 4 < \alpha \),

for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). If \( f : X \to Y \) is a mapping satisfying (3) for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \) with \( f(0) = 0 \), then there exists a unique quadratic-additive mapping \( F : X \to Y \) satisfying (7) for all \( x \in X \) and \( t > 0 \). Moreover if \( \alpha < 1 \) and \( N'_{\varphi(x_1,x_2,\ldots,x_n)} \) is continuous in \( x_1, x_2, \ldots, x_n \in X \setminus \{0\} \) under the condition (i), then \( f \) is a quadratic-additive mapping.

Now we have a general Hyers-Ulam stability results of the quadratic-additive functional equation (2) in the framework of normed spaces for \( n \) is odd.

**Corollary 4.3** Let \( X \) be a linear space, \( n \) be an odd number greater than 3, and \( Y \) be a complete normed-space and \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there is \( \varphi : (X \setminus \{0\})^n \to [0,\infty) \) such that

\[
\|Df(x_1,x_2,\ldots,x_n)\| \leq \varphi(x_1,x_2,\ldots,x_n)
\]
for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$. If $\varphi$ satisfies one of the following conditions:
(i) $\alpha \varphi(x_1, x_2, \ldots, x_n) \geq \varphi(2x, 2y, 2z, 2w)$ for some $0 < \alpha < 2$,
(ii) $\varphi(2x_1, 2x_2, \ldots, 2x_n) \geq \alpha \varphi(x_1, x_2, \ldots, x_n)$ for some $4 < \alpha$
for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, then there exists a unique mapping $F : X \to Y$ such that
\[
DF(x_1, x_2, \ldots, x_n) = 0
\]
for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and
\[
\|f(x) - F(x)\| \leq \begin{cases} 
\frac{2\Phi(x)}{(n-1)(2-\alpha)} & \text{if } \varphi \text{ satisfies (i)}, \\
\frac{2\Phi(x)}{(n-1)(\alpha-4)} & \text{if } \varphi \text{ satisfies (ii)} 
\end{cases}
\]
for all $x \in X \setminus \{0\}$, where $\Phi(x)$ is defined by
\[
\Phi(x) = \max(\varphi(\bar{x}), \varphi(-\bar{x})).
\]
Moreover, if $0 < \alpha < 1$ under the condition (i), then $f \equiv F$.

Now we have the Hyers-Ulam-Rassias stability of the quadratic-additive functional equation (2) in the framework of normed spaces.

**Corollary 4.4** Let $X$ be a normed space, $n$ be an odd number greater than 3, $p \in \mathbb{R} \setminus [1, 2]$, and $Y$ a complete normed-space. If $f : X \to Y$ is a mapping such that
\[
\|Df(x_1, x_2, \ldots, x_n)\| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p
\]
for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ with $f(0) = 0$, then there exists a unique mapping $F : X \to Y$ such that
\[
DF(x_1, x_2, \ldots, x_n) = 0
\]
for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and
\[
\|f(x) - F(x)\| \leq \begin{cases} 
0 & \text{if } p < 0, \\
\frac{2n\|x\|^p}{(n-1)(2-2p)} & \text{if } 0 \leq p < 1, \\
\frac{2n\|x\|^p}{(n-1)(2p-4)} & \text{if } 2 < p 
\end{cases}
\]
for all $x \in X \setminus \{0\}$.

**Proof.** If we denote by $\varphi(x_1, x_2, \ldots, x_n) = \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 4.1 with $\alpha = 2^p$. 
Corollary 4.5 Let $X$ be a normed space, $n$ be an odd number greater than 3, $p \in R\setminus[1,2]$ and $Y$ a complete normed-space. If $f : X \to Y$ is a mapping such that
\[\|Df(x_1, x_2, \cdots, x_n)\| \leq \sum_{i=1, x_i \neq 0}^{n} \|x_i\|^p\]
for all $x_1, x_2, \cdots, x_n \in X$ with $f(0) = 0$, then there exists a unique quadratic-additive mapping $F : X \to Y$ satisfying (8) for all $x \in X\setminus\{0\}$.

Proof. If we denote by $\varphi(x_1, x_2, \cdots, x_n) = \sum_{i=1, x_i \neq 0}^{n} \|x_i\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 4.2 with $\alpha = 2^p$.

References


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