On Uniform Convergence of Double Sine Series

under Condition of $p$-Supremum Bounded Variation Double Sequences

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Abstract

The classical theorem on the uniform convergence of sine series with monotone decreasing coefficients have been proved by Chaundy and Jollife in 1916. Recently, the monotone decreasing coefficients has been generalized by classes of Mean Bounded variation Sequences, Supremum Bounded Variation Sequence and $p$-Supremum Bounded Variation Sequences. In two variables, class of Mean Bounded Variation Double Sequences and Supremum Bounded Variation Double Sequences were studied under the uniform convergence of double sine series. We shall generalize those results by defining class of $p$-Supremum Bounded Variation Double Sequences and study uniform convergence of double sine series.
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1. Introduction

The classical theorem on the uniform convergence of sine series with monotone decreasing coefficients have been proved by Chaundy and Jollife in 1916 [1, 12] as stated in Theorem 1.1.

Theorem 1.1. Suppose that $\{a_k\} \subseteq [0,\infty)$ is decreasing tending to zero. The necessary and sufficient conditions for the uniform convergence of the series

$$\sum_{k=1}^{\infty} a_k \sin kx$$

is

$$\lim_{k \to \infty} k a_k = 0.$$ 

Many classes of sequences have been introduced by many researchers such as Tikhonov [11], Zhou [10], Korus [7]. These classes are more general than the classes of monotone decreasing coefficients (1.1). The definition of the classes of $SBVS$ (Supremum Bounded Variation Sequences), $SBVS$ (Supremum Bounded Variation Sequences), $SBVS2$ (Supremum Bounded Variation Sequences of second type) are the following.

$SBVS =$

$$\left\{ \{a_k\} \subseteq \mathbb{C} \mid \exists C > 0, \exists \gamma \geq 1 : \sum_{k=n}^{2n-1} |\Delta a_k| \leq \frac{C}{n^{m \geq [n/\gamma]}} \sum_{k=m}^{2m} |a_k|, n \geq \gamma \right\}$$

$SBVS2 =$

$$\left\{ \{a_k\} \subseteq \mathbb{C} \mid \exists C > 0, \exists \{b_k\} \subseteq \mathbb{C} : \sum_{k=n}^{2n-1} |\Delta a_k| \leq \frac{C}{n^{m \geq b(n)}} \sum_{k=m}^{2m} |a_k|, n \geq \gamma \right\}$$

where $\Delta a_k = a_k - a_{k+1}$ and $C$, $\lambda$, $\gamma$ depend only $\{a_k\}$, $[x]$ the greatest integer that is less than or equal $x$.

Imron et. al. [4] introduced classes of $SBVS_p$ (p-Supremum Bounded Variation Sequences) and $SBVS2_p$ (p-Supremum Bounded Variation Sequences of second type) stated in Definition1.2:

Definition 1.2. Let $\{a_n\}$ and $\beta = \{b_n\}$ be two sequences of complex and positive numbers, respectively. A couple $(a, \beta)$ is said to be

(i) $p$-Supremum Bounded Variation Sequences, written $(a, \beta) \in SBVS_p$, if there exist a positive constant $C$ and $\gamma \geq 1$ such that
Definition 1.3. A double sequence $a = \{a_{jk}\} \subseteq \mathbb{C}$ belongs to class $SBVSDS_1$, if there exists $C > 0$ and integer $\lambda \geq 2$ and $\{b_1(l), b_2(l), b_3(l)\}$, each one converges to infinity, all of them depend only $\{a_{jk}\}$ such that

$$
\sum_{j=m}^{2^{n-1}} |\Delta_{10} a_{jn}| \leq \frac{C}{m} \max_{b_1(m) \leq M} \sum_{j=M}^{2^M} |a_{jn}|, \quad m \geq \lambda, \quad n \geq 1,
$$

$$
\sum_{k=n}^{2^{m-1}} |\Delta_{01} a_{mk}| \leq \frac{C}{n} \max_{b_2(n) \leq N} \sum_{k=N}^{2^N} |a_{mk}|, \quad m \geq 1, \quad n \geq \lambda,
$$

$$
\sum_{j=m}^{2^{n-1}} \sum_{k=n}^{2^{m-1}} |\Delta_{11} a_{jk}| \leq \frac{C}{mn} \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2^M} \sum_{k=N}^{2^N} |a_{jk}|, \quad m, n \geq \lambda,
$$

where

\begin{align*}
\Delta_{10} a_{jk} &= a_{jk} - a_{j+1,k}, \\
\Delta_{01} a_{jk} &= a_{jk} - a_{j,k+1}, \\
\Delta_{11} a_{jk} &= \Delta_{10}(\Delta_{01} a_{jk}) = \Delta_{01}(\Delta_{10} a_{jk}) = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.
\end{align*}

Definition 1.4. A double sequence $a = \{a_{jk}\} \subseteq \mathbb{C}$ belongs to class $SBVSDS_2$, if there exists $C > 0$ and integer $\lambda \geq 1$ and $\{b(l)\}$, converging to infinity, depending only $\{a_{jk}\}$ such that

\begin{align*}
(\sum_{k=1}^{2^{n-1}} |a_k - a_{k+1}|^p)^{1/p} &\leq \frac{C}{n} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2^m} \beta_k \right), \quad n \geq 1, \\
(ii) - \text{Supremum Bounded Variation Sequences of second type, written,} & \\
(a, \beta) \in SBVSDS_2, & \text{if there exist a positive constant } C \text{ and } \{b(k)\} \subset [0, \infty) \text{ tending monotonically to infinity depending only on } \{a_k\}, \text{ such that} \\
(\sum_{k=1}^{2^{n-1}} |a_k - a_{k+1}|^p)^{1/p} &\leq \frac{C}{n} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2^m} \beta_k \right), \quad n \geq 1, \\
\text{for } 1 \leq p < \infty,
\end{align*}

As the single sine series (1.1) we have double sine series of two variables. Let $a = \{a_{jk}\} \subseteq \mathbb{C}$ and consider the double sine series of the form

$$
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin kx \quad (1.2)
$$

As an idea in one variable, to investigate the uniform convergence of double sine series, the coefficients of the series (1.2) are supposed to be member of class of general monotone double sequences. In two variables, Supremum Bounded Variation Double Sequences of first type ($SBVSDS_1$) and Supremum Bounded Variation Double Sequences of second type ($SBVSDS_2$) have been introduced by Korus [8].

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\[ \sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}| \leq \frac{c}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |a_{jn}|, \quad m \geq \lambda, \quad n \geq 1, \]
\[ \sum_{k=n}^{2n-1} |\Delta_{01} a_{mk}| \leq \frac{c}{n} \sup_{N \geq b(n)} \sum_{k=N}^{2N} |a_{mk}|, \quad m \geq 1, \quad n \geq \lambda, \]
\[ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Lambda_{11} a_{jk}| \leq \frac{c}{mn} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |a_{jk}|, \quad m, n \geq \lambda. \]

Related to double sine series of (1.2), Korus and Moricz [9] also consider the regular convergence of double sequence stated in Definition 1.5.

**Definition 1.5.** A double sequence \( a = \{a_{jk}\} \subseteq \mathbb{C} \) is regularly convergence if the sums

\[ \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \]

converge to a finite number as \( m \) and \( n \) tend to infinity independently of each other, moreover

\[ \sum_{j=1}^{\infty} a_{jn}, \quad n = 1, 2, 3, \ldots, \quad \text{and} \quad \sum_{k=1}^{\infty} a_{mk}, \quad m = 1, 2, 3, \ldots \]

are convergent.

In the present paper, we construct classes of \( p \)-Supremum Bounded Variation Double Sequences of first type \((SBVDS1_p)\) and \( p \)-Supremum Bounded Variation Double Sequences of second type \((SBVDS2_p)\) and investigate some of those two classes. Furthermore, we investigate the uniform regular convergence of the sine double series.

### 2. Uniform Convergence of Double Sine Series

In this section, we construct class of \( p \)-Supremum Bounded Variation Double Sequences First Type and Second Type. Furthermore we investigate the relations between those and also study other properties of class of \( p \)-Supremum Bounded Variation Double Sequences. The Goal of this paper is to study the uniform regular convergence of double sine series in Theorem 2.14.

**Definition 2.1.** Let \( a = \{a_{jk}\} \) and \( \beta = \{\beta_{jk}\} \) be two double sequences of complex and positive numbers, respectively. A couple \((a, \beta)\) is said to be class of \( p \)-Supremum Bounded Variation Double Sequences first type written \((a, \beta) \in SBVDS1_p\), if there exists \( C > 0 \) and integer \( \lambda \geq 2 \) and \( \{b_1(l)\}, \{b_2(l)\}, \{b_3(l)\} \), each one converges to infinity, all of them depend only \( \{a_{jk}\} \) such that

\[ (i) \quad \left( \sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}|^p \right)^{1/p} \leq \frac{c}{m} \max_{b_1(m) \leq a \leq b_2(m)} \sum_{j=M}^{2M} \beta_{jn}, \quad m \geq \lambda, \quad n \geq 1, \]
(ii) \( (\sum_{k=n}^{2n-1} |\Delta_1 a_{mk}|^p)^{1/p} \leq \frac{c}{m} \max_{b_2(n) \leq N \leq b_2(n)} \sum_{k=n}^{2N} \beta_{mk}, \ m \geq 1, \ n \geq \lambda, \)

(iii) \( (\sum_{j=m}^{2n-1} \sum_{k=n}^{2n-1} |\Delta_{11} a_{jk}|^p)^{1/p} \leq \frac{c}{mn} \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk}, m, n \geq \lambda, \)

for \( 1 \leq p < \infty. \)

**Definition 2.2.** Let \( a = \{a_{jk}\} \) and \( \beta = \{\beta_{jk}\} \) be two double sequences of complex and positive numbers, respectively. A couple \( (a, \beta) \) is said to be class of \( p \)-Supremum Bounded Variation Double Sequences second type written \( (a, \beta) \in SBVDS_2(p) \), if there exists \( C > 0 \) and integer \( \lambda \geq 1 \) and \( \{b(l)\} \), converging to infinity, depending only \( \{a_{jk}\} \) such that

(i) \( (\sum_{j=m}^{2m-1} |\Delta_1 a_{jn}|^p)^{1/p} \leq \frac{c}{m} \sup_{M \geq b(m)} \sum_{j=m}^{2M} \beta_{jn}, \ m \geq \lambda, \ n \geq 1, \)

(ii) \( (\sum_{k=n}^{2n-1} |\Delta_0 a_{mk}|^p)^{1/p} \leq \frac{c}{n} \sup_{N \geq b(n)} \sum_{k=n}^{2N} \beta_{mk}, \ m \geq 1, \ n \geq \lambda, \)

(iii) \( (\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} a_{jk}|^p)^{1/p} \leq \frac{c}{mn} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk}, m, n \geq \lambda, \)

for \( 1 \leq p < \infty. \)

**Theorem 2.3.** If \( 1 \leq p < \infty, \) then \( SBVDS_1(p) \subseteq SBVDS_2(p). \)

**Proof:**

First, we prove that \( SBVDS_1(p) \subseteq SBVDS_2(p), \) let \( (a, \beta) \in SBVDS_1(p) \) and put \( b(n) = \inf \{b_1(n), b_2(n), b_3(n), b_1(n+1), b_2(n+1), b_3(n+1), \ldots\} \), for \( 1 \leq p < \infty. \) From Definition 2.1 and Definition 2.2 \( (a, \beta) \in SBVDS_2(p), \) so we have \( SBVDS_1(p) \subseteq SBVDS_2(p). \)

The second, we prove that \( SBVDS_1(p) \supseteq SBVDS_2(p). \) From idea of example [8] can be constructed in the following way. Set \( m_r = 2(2^r) \) for \( r = 1, 2, 3, \ldots \) and

\[
a_{j1} = \begin{cases} 
0 & \text{if} \ 1 \leq j < m_1, \\
1 & \text{if} \ j = m_r, \\
0 & \text{if} \ n_j < j < m_r^2, \\
1 & \text{if} \ m_r^2 \leq j < 2m_r^2, \\
0 & \text{if} \ 2m_r^2 \leq j < m_{r+1}. 
\end{cases}
\]

We define the sequence \( \{\beta_{j1}\} \), where \( \beta_{j1} = a_{j1}, \) for every \( j. \) By Theorem 2.1. [6], \( (a_{j1}, \beta_{j1}) \in SBVDS_p \) with constant \( C = 4 \) and \( b(j) = \left\lfloor \frac{j}{2} \right\rfloor. \)
The second example

\[
a_{j_2} = \begin{cases}
0 & \text{if } 1 \leq j < m_1, \\
\frac{1}{r^2} & \text{if } j = m_r, \\
0 & \text{if } m_r < j < m_r^2, \\
\frac{1}{r^2} & \text{if } m_r^2 \leq j \leq 2m_r^2, \\
0 & \text{if } 2m_r^2 < j < m_{r+1}.
\end{cases}
\]

We define the sequence \( \{\beta_{j_2}\} \), where \( \beta_{j_2} = a_{j_2} \), for every \( j \). By Theorem 2.4. [6], \( \{a_{j_2}, \{\beta_{j_2}\}\} \in S\, B\, V\, D\, S_2p \) with constant \( C = 2 \) and \( b(j) = j^{1/2} \).

Finally, we define \( \{a_{jk}, \{\beta_{jk}\}\} \in S\, B\, V\, D\, S_2p \). Thus

\[
a_{jk} = 0 \quad \text{for} \quad j \geq 1, k \geq 3.
\]

It is clear that for \( n=2 \), by Definition 1.3 \( \{a_{jn}, \{\beta_{jn}\}\} \not\in S\, B\, V\, D\, S_1p \). Furthermore the second and third condition of Definition 1.4 are satisfied for \( C = 4 \) and \( b(l) = l^{1/2} \), this means \( \{a_{jn}, \{\beta_{jn}\}\} \in S\, B\, V\, D\, S_2p \). Thus \( S\, B\, V\, D\, S_1p \subset S\, B\, V\, D\, S_2p \). ■

**Theorem 2.4.** If \( 1 \leq p < q < \infty \), then \( S\, B\, V\, D\, S_1p \subset S\, B\, V\, D\, S_1q \).

**Proof:** Given \( (a, \beta) \in S\, B\, V\, D\, S_1p \) and for \( 1 \leq p < q < \infty \),

(i) From proof Theorem 3.1 [4] by replacing \( \Delta a_j \) to \( \Delta a_{jn} \) obtained

\[
\sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}|^{p/q} \leq \left( \sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}|^p \right)^{\frac{q}{p}},
\]

Hence by Definition 2.1 we have

\[
\left( \sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}|^q \right)^{1/q} \leq \left( \sum_{j=m}^{2m-1} |\Delta_{10} a_{jn}|^p \right)^{\frac{1}{p}} \leq \frac{C}{m} \max_{b_{1(n)} \leq \Delta b_1(m)} \sum_{j=M}^{2M} \beta_{jn}.
\]

(ii) Similarly by proof (i) and replacing \( \Delta a_{jn} \) to \( \Delta a_{mk} \) and Definition 2.1 we have

\[
\left( \sum_{k=n}^{2n-1} |\Delta_{01} a_{mk}|^q \right)^{1/q} \leq \left( \sum_{k=n}^{2n-1} |\Delta_{01} a_{mk}|^p \right)^{\frac{1}{p}} \leq \frac{C}{n} \max_{b_{1(n)} \leq \Delta b_1(n)} \sum_{k=N}^{2N} \beta_{mk}.
\]

(iii) Similarly by proof (i) and replacing \( \Delta a_{jn} \) to \( \Delta a_{jk} \) with double sumation and Definition 2.1 we have

\[
\left( \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} a_{jk}|^q \right)^{1/q} \leq \left( \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} a_{jk}|^p \right)^{\frac{1}{p}}
\]
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\[ \leq \frac{C}{mn} \sup_{m+n \geq b_2(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk} \]

From (i), (ii) and (iii), \((a, \beta) \in SBVDS_1 q\), then \(SBVDS_1 p \subseteq SBVDS_1 q\).

**Theorem 2.5.** If \(1 \leq p < q < \infty\), then \(SBVDS_2 p \subseteq SBVDS_2 q\).

**Proof:** Similarly as proof of Theorem 2.4 and Definition 2.2.

**Definition 2.6.** Let \(\beta = \{\beta_{jk}\}\) be double sequences of positive numbers.

(i) A class of \(p\)-Supremum Bounded Variation Double Sequences first type of \(\beta\), written \(SBVDS_1 p(\beta)\), is defined as
   \[\{a: (a, \beta) \in SBVDS_1 p\}\].

(ii) A class of \(p\)-Supremum Bounded Variation Double Sequences second type of \(\beta\), written \(SBVDS_2 p(\beta)\), is defined as
   \[\{a: (a, \beta) \in SBVDS_2 p\}\].

By Theorem 2.3, Theorem 2.4 and Theorem 2.5 and straight from Definition 2.6 we have Corollary 2.7, Corollary 2.8 and Corollary 2.9.

**Corollary 2.7.** If \(1 \leq p < \infty\), then \(SBVDS_1 p(\beta) \subseteq SBVDS_2 p(\beta)\).

**Corollary 2.8.** If \(1 \leq p < \infty\), then \(SBVDS_1 p(\beta) \subseteq SBVDS_2 p(\beta)\).

**Corollary 2.9.** If \(1 \leq p < \infty\), then \(SBVDS_1 p(\beta) \subseteq SBVDS_2 p(\beta)\).

Some properties of \(SBVDS_2 p\), \(1 \leq p < \infty\), are stated below. Theorem 2.10 give a sufficient condition for a couple of sequences in \(SBVDS_2 p\) to be have bounded variation introduced by Hardy [3]. The notion of bounded variation of double sequence of real or complex term is defined as follows:

The double sequence \(\{a_{nk}\}\) is bounded variation in \((n, k)\) if

(i) \(\{a_{nk}\}\) is for every fixed value of \(n\) and \(k\), bounded variation in \(n\) and \(k\).

(ii) the summation \(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\Delta a_{nk}|\) is convergent.

**Theorem 2.10.** Let \((a, \beta) \in SBVDS_2 p, 1 \leq p < \infty\). If

(i) for \(m \in \mathbb{N}\), \(m^{2-\frac{1}{p}} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn}\) decreasing monotone.

(ii) for \(n \in \mathbb{N}\), \(n^{2-\frac{1}{p}} \sup_{N \geq b(n)} \sum_{k=N}^{2N} \beta_{mk}\) decreasing monotone.

(iii) for \((m, n) \in \mathbb{N} \times \mathbb{N}\), \((mn)^{2-\frac{1}{p}} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk}\) decreasing monotone.
then

(i) \( m \sum_{j=m}^{\infty} |\Delta_{10} a_{jn}| \leq C \left( m^{2-1/p} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn} \right), \)

(ii) \( n \sum_{k=n}^{\infty} |\Delta_{01} a_{mk}| \leq C \left( n^{2-1/p} \sup_{N \geq b(n)} \sum_{k=N}^{2N} \beta_{mk} \right), \)

(iii) \( mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \leq C (mn)^{2-1/p} \left( \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk} \right), \)

holds respectively for \( p, 1 \leq p < \infty. \)

**Proof.** We prove this Theorem in three parts

(i) Let \( (a, b) \in SBVSD_{2p} \) and \( m^{2-1/p} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn} \)
decreasing monotone for \( m \in \mathbb{N} \). We denote

\[
m a_m = m (m^{-1/p}) \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn}
\]

for every \( m \in \mathbb{N} \). Therefore

\[
m \sum_{j=m}^{\infty} |\Delta_{10} a_{jn}| = m \sum_{j=0}^{\infty} \sum_{m=0}^{2^{s+1} m-1} |\Delta_{10} a_{jn}|^{1/p} \left( 2^s \right)^{1-1/p}
\]

\[
\leq m \sum_{j=0}^{\infty} \left( \frac{2^s}{2^{s+1}} \right)^{1/p} \left( \frac{2^{s+1} m-1}{2^{s+1} m} \right) \leq C' \sum_{j=0}^{\infty} \frac{2^{s+1} m-1}{2^{s+1} m} \leq C' \left( \frac{2^{s+1} m-1}{2^{s+1} m} \right)
\]

(ii) The proof of (ii) is similar with the proof of (i) by replacing \( m \) to \( n \) and \( \Delta_{10} a_{jn} \) to \( \Delta_{01} a_{mk} \).

(iii) Let \( (a, b) \in SBVSD_{2p} \) and \( (mn)^{2-1/p} \left( \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk} \right) \)
decreasing monotone for \( (m, n) \in \mathbb{N} \times \mathbb{N} \). We denote

\[
n \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| = mn \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{2^{s+1} m-1} \sum_{k=2^{s+1} m}^{2^{s+1} m-1} |\Delta_{11} a_{jk}|
\]
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\[
\leq mn \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \left( \sum_{j=2^{s}m}^{2^{s+1}m-1} \left| \Delta_{11} a_{jn} \right| \right)^{1/p} (2^{s}n)^{1-1/p} \tag{2.3}
\]
\[
\leq mn \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \left( \sum_{j=2^{s}m}^{2^{s+1}m-1} \sum_{k=2^{t}n}^{2^{t+1}n-1} \left| \Delta_{11} a_{jk} \right| \right)^{1/p} \tag{2.4}
\]

By Holder inequality obtained (2.3) and (2.4) obtained because of the decreasing monotone condition of Theorem 2.10 (iii).

The proof of this Theorem is complete. \]

**Theorem 2.11.** Let \((a, \beta) \in SBVDS_{2, p}, 1 \leq p < \infty. \) If

(i) the summation \( m^{2-1/p} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn}, \) tending to 0, for \( n \to \infty, \)

(ii) the summation \( n^{2-1/p} \sup_{N \geq b(n)} \sum_{k=0}^{2N} \beta_{mk}, \) tending to 0, for \( n \to \infty, \)

(iii) the summation \( (mn)^{2-1/p} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk}, \)

then, the sequence \( a = \{a_{jk}\} \) is bounded variation.

**Proof.**

(i) Given \((a, \beta) \in SBVDS_{2, p} \) and let

\[
m \alpha_{mn} = m \left( m^{1-1/p} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \beta_{jn} \right)
\]
tending to 0 for \( m \to \infty. \) We denote \( d_{m} = \sup_{v \geq m} (v \alpha_{vn}), \) for fixed \( n, \)
then \( d_{m} \to 0, \) for \( m \to \infty. \) For every \( \varepsilon > 0, \) there exists \( m_{0} \in \mathbb{N} \) such that \( |d_{m}| < \varepsilon \) for \( m \geq m_{0} \) and from the proof of Theorem 2.10 (i), we obtained

\[
m \sum_{s=0}^{\infty} |\Delta_{10} a_{jn}| \leq C \sum_{s=0}^{\infty} \frac{\alpha_{mn}}{2^{s}} \leq C \sum_{s=0}^{\infty} \frac{d_{m}}{2^{s}} \leq C d_{m}, \quad C = 2C'.
\]

Thus \( \sum_{j=m_{0}}^{\infty} |\Delta_{10} a_{jn}| \leq C \frac{d_{m}}{n_{0}} < C \frac{\varepsilon}{n_{0}} \)

Therefore \( \{a_{jn}\}_{j=1}^{\infty} \) is bounded variation.

(ii) Proof of (ii) similar with the proof of (i) by replacing \( m \) to \( n \) and \( \Delta_{10} a_{jn} \)
to \( \Delta_{01} a_{mk}. \) Therefore \( \{a_{mk}\}_{k=1}^{\infty} \) is bounded variation.
(iii) Given \((a,\beta) \in BVDS_{2p}\) and let

\[
mn\rho_{mn} = mn \left( (mn)^{1-\frac{1}{p}} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk} \right)
\]
tending to 0 for \(m + n \to \infty\). We denote

\[
d_{mn} = \sup_{s \geq m, t \geq n} (st\rho_{st}),
\]
then \(d_{mn} \to 0\) for \(m + n \to \infty\). For every \(0 < \varepsilon\), there exists 
\((m_0, n_0) \in \mathbb{N} \times \mathbb{N}\) such that \(|d_{mn}| < \varepsilon\) for \((m, n) \geq (m_0, n_0)\) and from the proof of Theorem 2.10 (iii), we obtained

\[
mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \leq C' \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(2^{s+t}mn)^{p-\varepsilon}t_{mn}}{2^{s+t}tmn}.
\]
Then we have

\[
\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \leq C' \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{(2^{s+t}mn)^{p-\varepsilon}t_{mn}}{2^{s+t}tmn}
\]

\[
\leq C' \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{d_{mn}}{2^{s+t}mn} \leq C \frac{d_{mn}}{mn}, \quad C = 4C'.
\]

Thus

\[
\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} a_{jk}| \leq C \frac{|d_{m_0n_0}|}{m_0n_0} \leq C \frac{\varepsilon}{m_0n_0}
\]
for \((m, n) \geq (m_0, n_0)\).

Therefore from proof (i), (ii) and (iii), the double sequence \(\{a_{jk}\}_{j,k=1}^{\infty}\) is bounded variation. The proof of this Theorem is complete. 

**Lemma 2.12.** Let \(a = \{a_{jk}\} \subseteq [0, \infty)\) and \((a,\beta) \in BVDS_{2p}\) \(1 \leq p < \infty\). If condition (i), (ii) and (iii) of Theorem 2.11. are satisfied, then

\(a_{jk} \to 0\) as \(j + k \to \infty\).

**Proof.** From idea proof of Lemma 3 [7], for \(t\) be arbitrary natural numbers and \(s = m + 1, m + 2, \ldots, 2m\), we have

\[
a_{mt} = \sum_{k=m}^{s-1} \Delta_{10} a_{kt} + a_{st} \leq \sum_{k=m}^{s-1} |\Delta_{10} a_{kt}| + a_{st}.
\]  

(2.5)

Next, an analogous argument, for \(s\) be arbitrary natural numbers and \(t = n + 1, n + 2, \ldots, 2n\), we have

\[
a_{sn} = \sum_{l=n}^{t-1} \Delta_{01} a_{sl} + a_{st} \leq \sum_{l=n}^{t-1} |\Delta_{01} a_{sl}| + a_{st}.
\]

(2.6)

Finally, let \(m, n\) be natural numbers. A double version of the above argument, for \(s = m + 1, m + 2, \ldots, 2m\) and \(t = n + 1, n + 2, \ldots, 2n\), we have

\[
a_{mn} = \sum_{k=m}^{s-1} \sum_{l=n}^{t-1} \Delta_{11} a_{kt} + a_{mt} + a_{sn} - a_{st}
\]
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\[ \leq \sum_{k=m}^{s-1} \sum_{l=n}^{t-1} |\Delta_{11}a_{kl}| + a_{mn} + a_{sn}. \quad (2.7) \]

If we add up all inequalities in (2.7) and from (2.5), (2.6), then by proof of Theorem 2.11, we have

\[ mn a_{mn} \leq mn \sum_{k=m}^{s-1} \sum_{l=n}^{t-1} |\Delta_{11}a_{kl}| + m \sum_{k=m}^{s-1} \Delta_{10}a_{sl} + n \sum_{k=m}^{t-1} \Delta_{01}a_{sk} \leq 3Ce \]

for \((m, n) \geq (m_0, n_0)\). Thus \(j_k a_{j_k} \to 0\) as \(j + k \to \infty\).

**Lemma 2.13.** If \((a, \beta) \in SBVDS_{2p}, 1 \leq p < \infty\) and condition (i), (ii) and (iii) of Theorem 2.11 are satisfied, then

(i) \(mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}a_{jk}| \to 0\),

(ii) \(mn \sum_{j=m}^{\infty} \text{sup}_{k \geq n} |\Delta_{10}a_{jn}| \to 0\),

(iii) \(mn \sum_{k=n}^{\infty} \text{sup}_{k \geq n} |\Delta_{01}a_{mk}| \to 0\).

**Proof.**

(i) From proof of Theorem 2.11. we have

\[ mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}a_{jk}| \leq Cd_{mn}, \]

for any \((m, n) \in \mathbb{N} \times \mathbb{N}\). By condition (iii) of Theorem 2.11, therefore

\[ mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}a_{jk}| \to 0, \text{ as } m + n \to \infty. \]

(ii) From Lemma 2.12, we obtained

\[ j_k |a_{j_k}| \to 0 \text{ as } j + k \to \infty. \]

Further

\[ \Delta_{10}a_{mn} = \sum_{k=n}^{\infty} \Delta_{11}a_{mk}, \Delta_{01}a_{mn} = \sum_{j=m}^{\infty} \Delta_{11}a_{jn}. \quad (2.8) \]

for \(m, n = 1, 2, 3, \ldots\). By (2.8) we have

\[ \text{sup}_{k \geq n} |\Delta_{10}a_{jk}| = \text{sup}_{k \geq n} |\sum_{s=k}^{\infty} \Delta_{11}a_{js}| \leq \sum_{k=n}^{\infty} |\Delta_{11}a_{jk}| \]

\(j, n = 1, 2, 3, \ldots\). Thus we have

\[ \sum_{j=m}^{\infty} \text{sup}_{k \geq n} |\Delta_{10}a_{jk}| \leq \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}a_{jk}| \]

and an application of (i) yields

\[ mn \sum_{j=m}^{\infty} \text{sup}_{k \geq n} |\Delta_{10}a_{jk}| \to 0. \]

(iii) A similar argument by replacing \(\Delta_{10}\) to \(\Delta_{01}\), obtained (iii).

**Theorem 2.14.** If \((a, \beta) \in SBVDS_{2p}, 1 \leq p < \infty\) and condition (i), (ii) and (iii) of Theorem 2.11 are satisfied, then the series \((1.2)\) converges uniformly, regularly at \((x, y)\) for all \(0 \leq x, y \leq \pi\).

**Proof.**

To prove the uniformly regularly convergent of double sine series (1.2),
first by letting the single sine series with coefficients of double sequence \( \{a_{jk}\} \)

\[
\sum_{j=1}^{\infty} a_{jn} \sin jx, n = 1, 2, 3, \ldots \tag{2.9}
\]

\[
\sum_{k=1}^{\infty} a_{mk} \sin ky, m = 1, 2, 3, \ldots \tag{2.10}
\]

Furthermore, by the condition Definition 2.2 (i) and (ii) it can be proved that for any \( m \geq 1, \ (\{a_{mk}\}_{k=1}^{\infty}, \beta) \in \mathcal{SBS}2_p \) and for any \( n \geq 1, \ (\{a_{jn}\}_{n=1}^{\infty}, \beta) \in \mathcal{SBS}2_p \). Hence by condition (i) and (ii) and Theorem 3.1. shown by Imron et. al [4] implies the uniform convergence of the series (2.9) and (2.10). The second, we show that the double series (1.2) is regularly convergence at \((x,y)\) for all \( 0 \leq x, y \leq \pi \).

(i) Following an idea from Moricz [2], for \((x, y) \neq (0, 0)\) we represent the rectangular partial sums

\[
S_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \sin jx \sin kx, \ m, n \geq 1, \tag{2.11}
\]

of series (1.2) to perform a double summation by part yields

\[
S_{mn}(x, y) = \sum_{j=1}^{m} \left( \sum_{k=1}^{n} D^*_j(x) D^*_k(y) \Delta_{11} a_{jk} + \sum_{j=1}^{m} D^*_j(x) D^*_n(y) \Delta_{10} a_{j,n+1} \right)
\]

\[
+ \sum_{k=1}^{n} D^*_m(x) D^*_k(y) \Delta_{01} a_{m+1,k} + a_{m+1,n+1} D^*_m(x) D^*_n(y) \tag{2.12}
\]

where \( D^*_n(x) \) conjugate Dirichlet kernel defined as

\[
D^*_n(x) = \sum_{k=1}^{n} \sin kx = \frac{\cos \frac{1}{2}x - \cos \frac{n+1}{2}x}{2 \sin \frac{1}{2}x}, n \geq 1
\]

and \(|D^*_n(x)| \leq \frac{\pi}{x} \) for \( x \in (0, \pi] \).

Given condition (iii), by Lemma 2.13 we obtained

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\Delta_{11} a_{jk}| < \infty
\]

thus

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |D^*_j(x) D^*_k(y) \Delta_{11} a_{jk}| < \infty \tag{2.13}
\]

Given condition (i), (ii) and (iii), by Lemma 2.12

\[
|a_{jk}| \to 0, \text{ for } j + k \to \infty \tag{2.14}
\]

and from (2.8) we have

\[
\Delta_{10} a_{j,n+1} = \sum_{k=n+1}^{\infty} \Delta_{11} a_{jk}.
\]

Further

\[
\sum_{j=1}^{\infty} \Delta_{10} a_{j,n+1} \leq \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} |\Delta_{11} a_{jk}| \to 0 \text{ as } n \to \infty.
\]

This implies that, for all \( 0 < x, y \leq \pi \)

\[
\sum_{j=1}^{m} D^*_j(x) D^*_n(y) \Delta_{10} a_{j,n+1} \to 0 \text{ as } n \to \infty,
\]
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uniformly in m. Similarly, for all \( 0 < x, y \leq \pi \)
\[
\sum_{k=1}^{n} D_m^*(x) D_k^*(y) \Delta_0 a_{m+1,k} \to 0 \text{ as } m \to \infty, \tag{2.15}
\]
uniformly in n. By (2.13)
\[
a_{m+1,n+1} D_m^*(x) D_n^*(y) \to 0 \text{ as } m + n \to \infty. \tag{2.16}
\]
Consequently by (2.11), (2.12), (2.14), (2.15) and (2.16)
\[
S_{mn}(x,y) \to 0, \text{ as } m + n \to \infty.
\]
(ii) For \((x,y) = (0,0)\), \(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin kx = 0\).

From the first and second part of this proof and by Definition 1.5 it can be concluded that series (1.2) is regular convergence. Furthermore by Lemma 2.12, Lemma 2.13 and Theorem 1 [6], the regular convergence is uniformly at \((x,y)\) for all \(0 \leq x, y \leq \pi\). The proof is complete. 

The uniform regular convergence of the series in (1.2) in class of \(SBVDS1_p\) for \(p, 1 \leq p < \infty\), is stated in Corollary 2.15. We abandon the proof, since it is similar to the proof of Theorem 2.14.

**Corollary 2.15.** Let \((a, \beta) \in SBVDS1_p\), \(1 \leq p < \infty\). If
(i) the summation \((m^{2^{-\frac{1}{p}} \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} \beta_{jn}} = o(1)\)
for \(m \to \infty\),
(ii) the summation \((n^{2^{-\frac{1}{p}} \max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} \beta_{mk}} = o(1)\)
for \(n \to \infty\),
(iii) the summation \((mn)^{2^{-\frac{1}{p}}(\sup_{M+N \geq \lambda b_2(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \beta_{jk}} = o(1)\)
for \(m + n \to \infty\),
then the series (1.2) is uniformly regularly convergence at \((x,y)\) for all \(0 \leq x, y \leq \pi\).

**3. Conclusions**

In this paper we have introduced the classes of \(SBVDS1_p\) and \(SBVDS2_p\). We have investigated that
(i) The class of \(SBVDS1_p\) and \(SBVDS2_p\) is more general than class \(SBDS1\) and class \(SBVDS2\) respectively introduced by Korus [8] straight from Definition 2.1 and Definition 2.2.
(ii) Uniformly regularly convergence of (1.2) in Theorem 2.14 is more general than Theorem 1 by Korus [8] and Theorem 1 by Korus and Moricz [9].
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