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The Homotopic Invariance for Fixed Points of Set-valued Mappings in Banach Spaces

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Abstract

We obtain the homotopic invariance for fixed points of set-valued contractions and nonexpansive mappings whose domain is a bounded nonexpansive retract and nonempty interior subset of a uniformly convex Banach space. This gives a partial answer to a problem posed by Sims, Xu and Yuan (2001).

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1 Introduction

Invariance under homotopy for fixed points of set-valued mappings in Banach spaces was first studied in 1996 by Frigon [3]. She obtained the homotopy results for set-valued mappings in uniformly convex Banach spaces. In 2001, Sims, Xu and Yuan [10] obtained the homotopic invariance for fixed points of set-valued mappings in another class of Banach spaces that satisfy Opial condition. For the homotopic invariance of set-valued mappings in metric spaces, Markin [9] and Dhompongsa et al. [2] obtain the results in hyperconvex metric spaces and CAT(0) spaces respectively. In [10] the authors point out that the proof of the main theorem of [3] (Theorem 4.6) contains a gap (on page 29, to ensure that the y belongs to the asymptotic center $A(\overline{U}, \{x_n\})$, one must show that y lies in $\overline{U}$). It therefore remains an open question whether or not the conclusion of Theorem 4.6 of [3] is true. In this paper, we obtain the homotopic invariance for fixed points of set-valued contractions and nonexpansive mappings whose domain is a bounded nonexpansive retract and nonempty interior subset of a uniformly convex Banach space. This gives a partial answer to the problem mentioned above.

2 Preliminaries

Let K be a nonempty subset of a Banach space X. We shall denote by ∂K the boundary of K in X, by $\text{int}(K)$ the interior of K in X, by $\mathcal{F}(K)$ the family of nonempty closed subsets of K, and by $\mathcal{K}(K)$ the family of nonempty compact subsets of K. Let $D(\cdot,\cdot)$ be the Hausdorff distance on $\mathcal{F}(X)$, i.e.,

$$D(A,B) := \max\left\{\sup_{a\in A} \text{dist}(a,B), \sup_{b\in B} \text{dist}(b,A)\right\}, \quad A,B \in \mathcal{F}(X),$$

where $\text{dist}(a,B) := \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the set B.

A set-valued mapping $T : K \to \mathcal{F}(X)$ is said to be a *contraction* if there exists a constant $\lambda \in [0,1)$ such that

$$D(T(x),T(y)) \leq \lambda\|x-y\|, \text{ for all } x,\, y \in K. \tag{1}$$

If (1) is valid when $\lambda = 1$, then T is called *nonexpansive*. A point x is called a fixed point of T if $x \in T(x)$. We shall denote by $Fix(T)$ the set of all fixed points of T.

Let $\{x_n\}$ be a bounded sequence in X, for $x \in X$ we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} \|x - x_n\|.$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Recall that a bounded sequence $\{x_n\}$ is called *regular* if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is welled known that every bounded sequence in X has a regular subsequence (see, e.g., [4] or [8]).

It is also known (see, e.g., [5], p. 167) that in a uniformly convex Banach space, $A(\{x_n\})$ consists of exactly one point.

Let $\mathcal{U}$ be a nontrivial ultrafilter on the natural numbers $\mathbf{N}$. Recall [1, 7] that the ultrapower $(X)_{\mathcal{U}}$ of a Banach space X is the quotient space of

$$\ell_\infty(X) = \Big\{\{x_n\} : x_n \in X \text{ for all } n \in \mathbf{N} \text{ and } \|\{x_n\}\| = \sup_n \|x_n\| < \infty\Big\}$$

by

$$\ker \mathcal{N} = \Big\{\{x_n\} \in \ell_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\Big\}.$$

One can prove that $\widetilde{X} = (X)_{\mathcal{U}}$ is a Banach space with the quotient norm given by $\|\{x_n\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|$, where $\{x_n\}_{\mathcal{U}}$ is the equivalence class of $\{x_n\}$. It is also clear that X is isometric to a subspace of $\widetilde{X}$ by the canonical embedding $x \to \{x, x, ...\}_{\mathcal{U}}$. If $K \subset X$, we shall use the symbols $\dot{K}$ and $\dot{x}$ to denote the images of K and x in $\widetilde{X}$ respectively and denote

$$\widetilde{K} = \Big\{\widetilde{x} \in \widetilde{X} : \exists \{x_n\} \text{ such that } \widetilde{x} = \{x_n\}_{\mathcal{U}} \text{ and } x_n \in K \text{ for all } n \in \mathbf{N}\Big\}.$$

Thus $\dot{x} = \{x, x, ...\}_{\mathcal{U}}$ and $\dot{K} = \{\dot{x} \in \widetilde{X} : x \in K\}$.

If $T : K \to \mathcal{F}(X)$ is a set-valued nonexpansive mapping, we define a set-valued mapping $\widetilde{T} : \widetilde{K} \to \mathcal{F}(\widetilde{X})$ by

$$\widetilde{T}(\widetilde{x}) := \Big\{\widetilde{u} \in \widetilde{X} : \exists \{u_n\} \text{ such that } \widetilde{u} = \{u_n\}_{\mathcal{U}} \text{ and } u_n \in T(x_n) \text{ for all } n \in \mathbf{N}\Big\},$$

where $\widetilde{x} = \{x_n\}_{\mathcal{U}} \in \widetilde{K}$. It is known from [2] and [11] that $\widetilde{T}$ is also nonexpansive.

3 A fixed point theorem

The following fact is a characterization of asymptotic centers. We omit the proof because it is similar to the one given in [2].

Proposition 3.1 *Let X be a uniformly convex Banach space and $\{x_n\}$ be a regular sequence in X. Then x is the asymptotic center of $\{x_n\}$ if and only if $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\widetilde{x} := \{x_n\}_{\mathcal{U}}$ in the ultrapower $\widetilde{X}$.*

In [10], Sims, Xu and Yuan obtain the homotopic invariance for fixed points of set-valued mappings in a Banach space that satisfy Opial condition. They base their results on the demiclosedness principle. They also mention that the demiclosedness principle for set-valued mappings in another class of uniformly convex Banach spaces is still unknown. The following fixed point theorem is very useful to our homotopy results which will be presented in the next sections. We also point out that the demiclosednees principle is not presented in our setting but an essential assumption is nonexpansive retraction.

Recall that a subset K of X is said to be a *retract* of X if there exists a continuous mapping $R : X \to K$ with $Fix(R) = K$. Any such mapping R is a *retraction* of X onto K. If R is nonexpansive, then K is said to be a *nonexpansive retract* of X.

Theorem 3.2 *Let K be a nonempty subset of a uniformly convex Banach space X, $R : X \to K$ be a nonexpansive retraction and $T : K \to \mathcal{K}(X)$ be a nonexpansive mapping. Suppose that $dist(x_n, T(x_n)) \to 0$ for some bounded sequence $\{x_n\} \subset K$. Then T has a fixed point in K.*

Proof. By passing to a subsequence we may assume that $\{x_n\}$ is regular. Let x be the asymptotic center of $\{x_n\}$. By Proposition 3.1, $\dot{x}$ is the unique point of $\dot{X}$ which is nearest to $\widetilde{x} := \{x_n\}_{\mathcal{U}}$. Since R is nonexpansive, $r(R(x), \{x_n\}) \le r(x, \{x_n\})$ and hence $x = R(x) \in K$ by the uniqueness of asymptotic centers. Thus $\dot{x} \in \dot{K}$. Since $\widetilde{x} \in \widetilde{T}(\widetilde{x})$ and $\widetilde{T}$ is nonexpansive,

$$\text{dist}(\widetilde{x}, \widetilde{T}(\dot{x})) \le D\left(\widetilde{T}(\widetilde{x}), \widetilde{T}(\dot{x})\right) \le \|\widetilde{x} - \dot{x}\|.$$

Since $T(x)$ is compact, it follows from [6, Proposition 1] that $\widetilde{T}(\dot{x}) = \dot{T(x)}$. It is also implies that $\text{dist}(\widetilde{x}, \widetilde{T}(\dot{x})) = \|\widetilde{x} - \dot{u}\|$ for some $\dot{u} \in \widetilde{T}(\dot{x})$. Then by the uniqueness of $\dot{x}$, we have $\dot{x} = \dot{u} \in \dot{T(x)}$, this in turn implies $x \in T(x)$ ■

4 Homotopic invariance for set-valued contractions

Definition 4.1 Let K be a nonempty subset of a Banach space X. A set-valued mapping $H : [0,1] \times K \to \mathcal{F}(X)$ is said to be *equi-continuous* in $t \in [0,1]$ over $x \in K$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $t, s \in [0,1]$ with $|t - s| < \delta$, we have $D(H(t,x), H(s,x)) < \varepsilon$ for all $x \in K$.

The following result is an analog of Theorem 3.1 of [10].

Theorem 4.2 *Let K be a nonexpansive retract subset of a uniformly convex Banach space X with $int(K) \neq \emptyset$, let $\{T_t\}_{0 \le t \le 1}$ be a family of λ-contractions from K to $\mathcal{K}(X)$ which is equi-continuous in $t \in [0,1]$ over $x \in K$. Assume that some T_t has a fixed point in K, and assume that every T_t is fixed point free on ∂K. Then T_t has a fixed point in K for each $t \in [0,1]$.*

Proof. Let $V = \{t \in [0,1] : T_t$ has a fixed point in $K\}$. Then V is nonempty by assumption. We show that V is both open and closed in $[0,1]$ and therefore conclude that $V = [0,1]$. The proof that V is open in $[0,1]$ is identical to the one given in the proof of Lemma 3.1 of [10]. To show that V is closed, we assume $\{t_n\} \subset V$ is such that $t_n \to t_0$. Then for each n, there exists $x_n \in K$ such that $x_n \in T_{t_n}(x_n)$. We note that $\{x_n\}$ is bounded (the proof is similar to the one given in Theorem 3.1 of [10]). By equi-continuity of T_{t_0} we have

$$\operatorname{dist}(x_n, T_{t_0}(x_n)) \le D\left(T_{t_n}(x_n), T_{t_0}(x_n)\right) \longrightarrow 0 \text{ as } n \to \infty.$$

By Theorem 3.2, T_{t_0} has a fixed point in K, so $t_0 \in V$ as desired. ∎

By applying Theorem 4.2, we can obtain the following corollary.

Corollary 4.3 *Let K be a nonexpansive retract subset of a uniformly convex Banach space X with int$(K) \neq \emptyset$. Suppose $T, G : K \to \mathcal{K}(X)$ are two set-valued $\lambda-$contraction mappings and suppose $H : [0,1] \times K \to \mathcal{K}(X)$ is a homotopy satisfying*

(1) *$H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$;*

(2) *for each $t \in [0,1]$, $H(t,\cdot)$ is a set-valued $\lambda-$contraction mapping ;*

(3) *$H(t,x)$ is equi-continuous in $t \in [0,1]$ over $x \in K$;*

(4) *for each $t \in [0,1]$, $H(t,\cdot)$ is fixed point free on ∂K.*

Then T has a fixed point in K if and only if G has a fixed point in K.

5 Homotopic invariance for set-valued nonexpansive mappings

We present now the homotopic invariance for set-valued nonexpansive mappings.

Theorem 5.1 *Let K be a bounded nonexpansive retract subset of a uniformly convex Banach space X with int$(K) \neq \emptyset$. Suppose $T, G : K \to \mathcal{K}(X)$ are two set-valued nonexpansive mappings and suppose there exists a homotopy $H : [0,1] \times K \to \mathcal{K}(X)$ such that*

(1) *$H(0,\cdot) = T(\cdot)$ and $H(1,\cdot) = G(\cdot)$;*

(2) *for each $t \in [0,1]$, $H(t,\cdot)$ is a set-valued nonexpansive mapping from K to $\mathcal{K}(X)$;*

(3) *$H(t,x)$ is equi-continuous in $t \in [0,1]$ over K ;*

(4) *for each sequence* (t_n) *in* $[0,1]$ *with*

$$\inf_{x\in K} dist\,(x, H\,(t_n, x)) > 0,$$

$\lim_{n\to\infty} t_n = t_0$ *implies* $\inf_{x\in K} dist\,(x, H\,(t_0, x)) > 0.$

Then T has a fixed point in K if and only if G has a fixed point in K.

Proof. Assume that T has a fixed point in K, and let

$$V = \{t \in [0,1] : \text{ there exists } x \in K \text{ such that } x \in H\,(t, x)\}.$$

We can show that V is closed as in the proof of Theorem 4.2. Suppose that V is not open. Then there exists $t_0 \in V$ and a sequence $\{t_n\} \subset [0,1]\setminus V$ such that $\lim_{n\to\infty} t_n = t_0$. Since $t_n \notin V$, $\text{dist}(x, H\,(t_n, x)) > 0$ for all $n \in \mathbf{N}$ and $x \in K$. We claim that

$$\inf_{x\in K} dist\,(x, H\,(t_n, x)) > 0 \text{ for all } n \in \mathbf{N}.$$

Otherwise, there exists a sequence $\{x_m\} \subset K$ such that

$$\lim_{m\to\infty} dist\,(x_m, H\,(t_n, x_m)) = 0.$$

Then by Theorem 3.2, $H\,(t_n, \cdot)$ has a fixed point in K. But this contradicts to $t_n \notin V$, so we have the claim. Condition (4) implies

$$\inf_{x\in K} dist\,(x, H\,(t_0, x)) > 0,$$

which in turn implies $t_0 \notin V$ and this is a contradiction. Therefore V is open in $[0,1]$, and hence $V = [0,1]$, from which the conclusion follows. ■

If we put some additional conditions on the homotopy H, we can drop the assumption (4) in Theorem 5.1.

Theorem 5.2 *Let K be a bounded nonexpansive retract subset of a uniformly convex Banach space X with int$(K) \neq \emptyset$. Let $F : K \to \mathcal{K}\,(X)$ be a nonexpansive mapping and there exists a set-valued homotopy $H : [0,1]\times K \to \mathcal{K}\,(X)$ such that*

(1) $H\,(1, \cdot) = F\,(\cdot)$;

(2) $H(0, \cdot)$ *has a fixed point in* K ;

(3) *for each* $t \in [0,1)$, *we have a* $\lambda_t \in [0,1)$ *such that* $H(s, \cdot)$ *is a* λ_t-*contraction for all* $s \in [0,t]$;

(4) $H\,(t, x)$ *is equi-continuous in* $t \in [0,1]$ *over* $x \in K$;

(5) *for each* $t \in [0,1)$, $H(t, \cdot)$ *is fixed point free on* ∂K.

Then F has a fixed point.

Proof. Apply Theorem 4.2 to the homotopy H restricted to $[0,t] \times K$ to get a fixed point of $H(s,\cdot)$ for each $s \in [0,t]$. Taking $t_n \to 1$ yields a sequence $\{x_n\}$ in K such that $x_n \in H(t_n, x_n)$ for all n. By the equi-continuity of H, we have

$$\operatorname{dist}(x_n, F(x_n)) \leq D(H(t_n, x_n), H(1, x_n)) \longrightarrow 0 \text{ as } n \to \infty.$$

By Theorem 3.2, F has a fixed point in K. ■

As a corollary, we obtain the following alternative priciple.

Corollary 5.3 (Nonlinear Alternative) *Let X, K and F be as in Theorem 5.2 and assume that the origin is contained in the interior of K. Then either F has a fixed point, or there are some $x \in \partial K$ and some $t \in (0,1)$ such that $x \in tF(x)$.*

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