Volume-Preserving Diffeomorphisms with Periodic Shadowing

Manseob Lee

Department of Mathematics, Mokwon University
Daejeon, 302-729, Korea
lmsds@mokwon.ac.kr

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Abstract

We show that if a volume-preserving diffeomorphism belongs to the \( C^1 \)-interior of the set of all volume preserving diffeomorphisms having the periodic shadowing property then it is Anosov.

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1 Introduction

The shadowing theory is closely related to the stability condition (see [8, 9]). We will study a kind of the shadowing property which is called periodic shadowing property. The notion of the periodic shadowing property was well studied by [7]. They showed that the following.

**Theorem 1.1** [7, Theorem] Let \( f \in \text{Diff}(M) \). The following are equivalent;

(a) \( f \) has the Lipschitz periodic shadowing property.

(b) \( f \) belongs to the \( C^1 \)-interior of the set of diffeomorphisms having the periodic shadowing property.

(c) \( f \) satisfies both Axiom A and the no-cycle condition.
Let $M$ be a $d$-dimensional ($d \geq 2$) Riemannian closed and connected manifold and let $d(\cdot, \cdot)$ denotes the distance on $M$ inherited by the Riemannian structure. We endow $M$ with a volume-form (cf. [5]) and let $\mu$ denote the Lebesgue measure related to it. Let $\text{Diff}_1^\mu(M)$ denote the set of volume-preserving diffeomorphisms defined on $M$, i.e. those diffeomorphisms such that $\mu(B) = \mu(f(B))$ for any $\mu$-measurable subset $B$. Consider this space endowed with the $C^1$ Whitney topology. The Riemannian inner-product induces a norm $\| \cdot \|$ on the tangent bundle $T_x M$. We will use the usual uniform norm of a bounded linear map $A$ given by $\|A\| = \sup_{\|v\|=1} \|Av\|$. Let $f \in \text{Diff}_1^\mu(M)$. Given $\delta > 0$, we say that a sequence of points $\{x_i\}_{i \in \mathbb{Z}} \subset M$ is a $\delta$-pseudo orbit of $f$ if $d(f(x_i), x_i) < \delta$ for all $i \in \mathbb{Z}$. We say that a $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ is a $\delta$-periodic pseudo orbit if $x_{n+i} = x_i$ for some $n \in \mathbb{Z}$. We say that $f$ has the periodic shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that for any periodic $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ with $x_{n+i} = x_i$, there is $y \in P(f)$ such that $d(f^n(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

Let $\Lambda$ be a closed $f$-invariant set. We say that $\Lambda$ is hyperbolic if the tangent bundle $T_\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$
\|D_x f^n|_{E^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u}\| \leq C\lambda^n
$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then $f$ is Anosov.

We say that $f$ has the $C^1$-robustly periodic shadowing property if there is a $C^1$-neighborhood $U(f) \subset \text{Diff}_1^\mu(M)$ of $f$ such that for any $g \in U(f)$, $g$ has the periodic shadowing property. In [2], Bessa proved that if $f$ has the $C^1$-robustly shadowing property then it is Anosov. Bessa, Lee and Wen shown in [3] that if $f$ has the $C^1$-robustly specification property then it is Anosov, and $f$ is $C^1$-robustly expansive then it is Anosov. From the results, we study the $C^1$-robustly periodic shadowing property. Then we have

**Theorem 1.2** Let $f \in \text{Diff}_1^\mu(M)$. The following are equivalent:

(a) $f$ has the $C^1$-robustly periodic shadowing property,

(b) $f$ is Anosov.

## 2 Proof of Theorem 1.2

Let $M$ be as before, and let $f \in \text{Diff}_1^\mu(M)$. To prove, we will use the following version of the Franks’ lemma for the conservative case which is stated and proved in [4, Proposition 7.4].

**Lemma 2.1** Let $f \in \text{Diff}_1^\mu(M)$, and $U(f)$ be a $C^1$-neighborhood of $f$ in $\text{Diff}_1^\mu(M)$. Then there exist a $C^1$-neighborhood $U_0(f) \subset U(f)$ of $f$ and $\epsilon > 0$ such that
if $g \in U_0(f)$, any finite $f$-invariant set $E = \{x_1, \ldots, x_m\}$, any neighborhood $U$ of $E$ and any volume-preserving linear maps $L_j : T_{x_j}M \to T_{g(x_j)}M$ with $\|L_j - D_{x_j}g\| \leq \epsilon$ for all $j = 1, \ldots, m$, there is a conservative diffeomorphism $g_1 \in U(f)$ coinciding with $f$ on $E$ and out of $U$, and $D_{x_j}g_1 = L_j$ for all $j = 1, \ldots, m$.

In the volume preserving case, the Axiom A condition is equivalent to the diffeomorphism be Anosov, since $\Omega(f) = M$ by Poincaré Recurrence Theorem. We define the set $F_\mu(M)$ as the set of diffeomorphisms $f \in \text{Diff}_\mu(M)$ which has a $C^1$-neighborhood $U(f) \subset \text{Diff}_\mu(M)$ such that if for any $g \in U(f)$, every periodic point of $g$ is hyperbolic. Note that $F_\mu(M) \subset \mathcal{F}(M)$ (see [1, Corollary 1.2]). Very recently, Arbieto and Catalan [1] proved that if a volume preserving diffeomorphism is contained in $F_\mu(M)$ then it is Anosov. We can restate as follows.

**Theorem 2.2** [1, Theorem 1.1] If $f \in F_\mu(M)$ then $f$ is Anosov.

To prove Theorem 1.2, it is enough to show that $f \in F_\mu(M)$.

**Remark 2.3** From the Moser’s Theorem (see [5]), there is a smooth conservative change of coordinates $\varphi_x : U(x) \to T_xM$ such that $\varphi_x(x) = \overline{0}$, where $U(x)$ is a small neighborhood of $x \in M$.

**Lemma 2.4** Let $f \in \text{Diff}_\mu(M)$. If $f$ has the $C^1$-robustly periodic shadowing property, then $f \in F_\mu(M)$.

**Proof.** Suppose that $f$ has the $C^1$-robustly periodic shadowing property. Let $U(f) \subset \text{Diff}_\mu(M)$ be a $C^1$-neighborhood of $f$. Then for any $g \in U(f)$, $g$ has the periodic shadowing property. To derive a contradiction, we may assume that $f \notin F_\mu(M)$. Then there is a nonhyperbolic periodic point $p \in P(g)$ for some $g \in U(f)$. For simplicity, we assume that $g(p) = p$. Then there is an eigenvalue $\lambda$ of $D_pg$ such that $|\lambda| = 1$, and $T_pM = E_p^s \oplus E_p^u \oplus E_p^c$, where $E_p^s$ is the eigenspace corresponding to the eigenvalues of the smaller than 1, and $E_p^u$ is the eigenspace corresponding to the eigenvalues of the greater than 1, and $E_p^c$ the eigenspace corresponding to $\lambda$. Then we see that if $\lambda \in \mathbb{R}$ then $\dim E_p^c = 1$, and if $\lambda \in \mathbb{C}$ then $\dim E_p^c = 2$.

First, we consider $\dim E_p^c = 1$. For simplicity, we may assume that $\lambda = 1$ (the other case is similar). By making use of the Lemma 2.1, we linearize $g$ at $p$ with respect to Moser’s Theorem; that is, by choosing $\alpha > 0$ sufficiently small we construct $g_1 C^1$-nearby $g$ such that

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_pg \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$
Then \( g_1(p) = g(p) = p \). Since the eigenvalue \( \lambda \) of \( D_p g_1 \) is one, \( D_p g_1(v) = v \) for any \( v \in E^s_p(\alpha) \). Take \( v_0 \in E^s_p(\alpha) \) such that \( \|v_0\| = \alpha/4 \). We set

\[
J_{v_0} = \{ t \cdot v_0 : -1 \leq t \leq 1 \} \subset \varphi_p(B_{\alpha}(p)),
\]

and \( \varphi_p^{-1}(J_{v_0}) = J_p \). Since \( g_1(J_p) = J_p \) is the identity map, \( \varphi_p^{-1}(J_{v_0}) = J_p \) is \( g_1 \)-invariant. Take \( \epsilon = \alpha/8 \). Let \( 0 < \delta = \delta(\epsilon) < \epsilon \) be the number of the periodic shadowing property of \( g_1 \). Take \( x_0 = x, x_1, \ldots, x_m \in J_p \) such that \( d(x_i, x_j) < \delta \) for all \( 0 \leq i \neq j \leq m \), and \( d(x_i, x_m) = 4\epsilon \). We have \( \xi_1 = \{ x(=x_0), x_1, \ldots, x_m, x_{m-1}, \ldots, x \} \) is a finite \( \delta \)-2m-periodic pseudo orbit of \( g_1 \). Then \( \xi = \{ \ldots, \xi_1, \xi_1, \ldots \} = \{ x_i \}_{i \in \mathbb{Z}} \) is \( \delta \)-periodic pseudo orbit of \( g_1 \) and it is clear \( \xi \in J_p \). By the periodic shadowing property, there is a periodic shadowing point \( y \in P(g_1) \) such that \( d(g_1(y), x_i) < \epsilon \) for all \( i \in \mathbb{Z} \). If \( y \in P(g_1) \setminus J_p \) then by Moser’s Theorem, \( y = \varphi_p^{-1}(w) = \varphi_p^{-1}(w^s, w^u, w^c) \), where \( w = (w^s, w^u, w^c) \in E^s_p \oplus E^u_p \oplus E^c_p \). Since \( g_1 : J_p \to J_p \) is the identity map,

\[
g_1^{j}(\varphi_p^{-1}(0, 0, w^c)) = \varphi_p^{-1}(0, 0, w^c) \in J_p
\]

for all \( i \in \mathbb{Z} \). Thus one can find \( k \in \mathbb{Z} \) such that

\[
d(g_1^{k}(\varphi_p^{-1}(w^s, w^u, 0)), g_1^{k}(\varphi_p^{-1}(0, 0, w^c))) = d(g_1^{k}(\varphi_p^{-1}(w^s, w^u, 0)), \varphi_p^{-1}(0, 0, w^c)) \geq \epsilon.
\]

This is a contradiction by the periodic shadowing property. Thus the periodic shadowing point have to be in \( J_p \). But, \( g_1 : J_p \to J_p \) is the identity map, for every point \( y \in J_p \) is the fixed point of \( g_1 \). Thus \( d(g_1^{k}(y), x_i) = d(y, x_i) < \epsilon \) for all \( i \in \mathbb{Z} \). Since \( d(x_0, x_m) = 2\epsilon \) there is \( k > 0 \) such that \( d(g_1^{k}(y), x_k) = d(y, x_k) > \epsilon \). This is a contradiction by the periodic shadowing property.

Finally, if \( \lambda \in \mathbb{C} \), then \( \dim E^c_p = 2 \). To avoid the notational complexity, we may assume that \( g(p) = p \). As in the first case, by Lemma 2.1, there are \( \alpha > 0 \) and \( g_1 \in \mathcal{V}(f) \) such that \( g_1(p) = g(p) = p \) and

\[
g_1(x) = \begin{cases} 
\varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_{\alpha}(p), \\
g(x) & \text{if } x \notin B_{4\alpha}(p).
\end{cases}
\]

With a \( C^1 \)-small modification of the map \( D_p g \), we may suppose that there is \( l > 0 \) (the minimum number) such that \( D_p g^i(v) = v \) for any \( v \in \varphi_p(B_{\alpha}(p)) \subset T_p M \). Take \( v_0 \in \varphi_p(B_{\alpha}(p)) \) such that \( \|v_0\| = \alpha/4 \), and set

\[
\mathcal{L}_p = \varphi_p^{-1}(\{ t \cdot v_0 : 1 \leq t \leq 1 + \alpha/4 \}).
\]

Then \( \mathcal{L}_p \) is an arc such that (a) \( g_1^{j}(\mathcal{L}_p) \cap g_1^{j}(\mathcal{L}_p) = \emptyset \) for \( 0 \leq i \neq j \leq l - 1 \), (b) \( g_1^{j}(\mathcal{L}_p) = \mathcal{L}_p \), and (c) \( g_1^{j}|_{\mathcal{L}_p} \) is the identity map. Note that \( g_1 \) has the periodic shadowing property if and only if \( g_1^{k} \) has the periodic shadowing property, for all \( k \in \mathbb{Z} \). As in the first case, we can show that \( g_1 \) does not have the periodic shadowing property, which contradicts the fact that \( g_1 \in \mathcal{U}(f) \). Thus,
if $f$ belongs to the $C^1$-interior of the set of a volume preserving diffeomorphism having the periodic shadowing property, every periodic point of $f$ is hyperbolic.

□

**Proof of Theorem 1.2.** Let $f \in \text{Diff}_\mu(M)$ has the $C^1$-robustly periodic shadowing property. Then by Lemma 2.4, $f \in \mathcal{F}_\mu(M)$. By Theorem 2.2, $f$ is Anosov. □

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**References**


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