A Hyperbolic Uniform Spline Method for Linear Fourth Order Boundary Value Problem

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Abstract

In this paper, a numerical method is developed for solving a linear fourth order boundary value problem (4VBP) by using the hyperbolic uniform spline of order 3 (lower order). There are both proved to be first-order convergent. Numerical results confirm the order of convergence predicted by the analysis.

Mathematics Subject Classification: 41A05, 41A15, 43A90, 65D05

Keywords: Boundary value problem, hyperbolic uniform spline of order
1 Introduction

In recent years, much attention have been given to solve the fourth-order boundary value problems, which have application in various branches of applied sciences [4]. A spline has been widely applied for the numerical solutions of some ordinary and partial differential equations in the numerical analysis. Many authors have used numerical and approximate methods to solve third and fourth-order BVPs. Some of the details about the numerical methods can be found in references [1, 9, 10]. In a series of paper by Caglar et al. [2, 3] BVPs of order two, third, fourth and fifth were solved using third, fourth and sixth-degree splines. Lamnii et al. [6, 7] discussed boundary-value problems based on spline interpolation and quasi-interpolation with second order convergence. The numerical analysis literature contains few on the solution of boundary value problems by using the hyperbolic B-splines. Literature proposes several approaches based on polynomial splines with higher degrees, which affect the computational efficiency in practical applications to solving boundary value problems are all polynomial splines. This motivates us to use hyperbolic B-splines of order 3 (lower order) to solve these problems. In this paper we study a method based on the hyperbolic B-splines of order 3 for constructing numerical solutions to fourth-order boundary value problems (4BVPs) of the form:

\[ y^{(4)}(\theta) + f(\theta)y(\theta) = g(\theta), \]

with boundary conditions:

\[ y(a) = a_0, \quad y'(a) = a_1, \quad y(b) = b_0, \quad y'(b) = b_1, \]

where \( f(\theta) \) and \( g(\theta) \) are given continuous functions defined in the bounded interval \([a, b]\), \( a_i(i = 0, 1) \), and \( b_i(i = 0, 1) \) are real constants.

The rest of paper is organized as follows. In Section 2, we give an explicit representation of B-splines of order 3. The interpolation hyperbolic B-splines is developed in Section 3. Solution and the convergence analysis is presented in Section 4. Numerical examples are presented in Section 5. Finally, we give conclusion in Section 6.

2 Hyperbolic B-splines of order 3

In this section, we briefly give an explicit representation of B-splines of order 3 and we give the interesting properties of UAH B-splines of order 3, for more details see [8]. To do this, we need the following notations. Suppose \( k \) and
interger such that $k \geq 1$. Let $m_k = 2^k$ and $h_k = \frac{b-a}{m_k-2}$. Put

$$
\begin{aligned}
\theta_{k-2}^k &= \theta_{k-1}^k = \theta_0^k = a, \\
\theta_i^k &= a + ih_k, i = 1...m_k - 3, \\
\theta_{m_k-2}^k &= \theta_{m_k-1}^k = \theta_{m_k}^k = b,
\end{aligned}
$$

(3)

the set of knots that subdivide the interval $I = [a, b]$ uniformly. The hyperbolic tension splines space of order 3 is defined as follows

$$
\mathcal{V}_k = \{s \in C^3(I) : s|_{[\theta_i^k, \theta_{i+1}^k]} \in \Gamma_3\text{ where } \Gamma_3 = \{1, \cosh(\theta), \sinh(\theta)\}. 
$$

The dimension of $\mathcal{V}_k$ is $m_k$ and the third-order hyperbolic B-splines are given by:

for $i = 0, 1, ..., m_k - 5,$

$$
\nu_{i,k}(\theta) = C_k \begin{cases}
2 \sinh(\frac{\theta - \theta_i^k}{2})^2, & \theta_i^k \leq \theta < \theta_{i+1}^k; \\
2 \cosh(h_k) - \cosh(\theta_{i+1}^k - \theta) - \cosh(\theta - \theta_{i+2}^k), & \theta_{i+1}^k \leq \theta < \theta_{i+2}^k; \\
2 \sinh(\frac{\theta_{i+3}^k - \theta}{2})^2, & \theta_{i+2}^k \leq \theta < \theta_{i+3}^k; \\
0, & \text{otherwise.}
\end{cases}
$$

with the respective left and right hand side boundary hyperbolic B-splines are

$$
\nu_{-2,k}(\theta) = C_k \begin{cases}
4 \sinh(\frac{-\theta + a + h_k}{2})^2, & \theta_0^k \leq \theta < \theta_1^k \\
0, & \text{otherwise.}
\end{cases}
$$

$$
\nu_{-1,k}(\theta) = C_k \begin{cases}
1 + 2 \cosh(h_k) - 2 \cosh(h_k - \theta + a) - \cosh(-a + \theta), & \theta_0^k \leq \theta < \theta_1^k \\
2 \sinh(\frac{a + 2h_k - \theta}{2})^2, & \theta_1^k \leq \theta < \theta_2^k \\
0, & \text{otherwise.}
\end{cases}
$$

$$
\nu_{m_k-4,k}(\theta) = C_k \begin{cases}
2 \sinh(\frac{a + 2h_k - b}{2})^2, & \theta_{m_k-4}^k \leq \theta < \theta_{m_k-3}^k \\
1 + 2 \cosh(h_k) - \cosh(b - \theta) - 2 \cosh(\theta + h_k - b), & \theta_{m_k-3}^k \leq \theta < \theta_{m_k-2}^k \\
0, & \text{otherwise.}
\end{cases}
$$

$$
\nu_{m_k-3,k}(\theta) = C_k \begin{cases}
4 \sinh(\frac{\theta + h_k - b}{2})^2, & \theta_{m_k-3}^k \leq \theta < \theta_{m_k-2}^k \\
0, & \text{otherwise.}
\end{cases}
$$

where $C_k = \frac{1}{4 \sinh(\frac{\theta_k^k}{2})^2}$.

The hyperbolic B-splines of order 3 possess all the desirable properties of classical polynomial B-splines, see [8]. In this paper, we limit ourselves to list some of them

- $\nu_{i,k}(\theta)$ is supported on the interval $[\theta_{i+1}^k, \theta_{i+1}^k]$;
- Positivity : $\nu_{i,k}(\theta) \geq 0, \forall \theta \in [\theta_{i+1}^k, \theta_{i+1}^k]$;
- Partition of unity: $\sum_{i=-2}^{m_k-3} \nu_{i,k}(\theta) = 1$. 
Table 1: Values of $\nu_{i,k}(\theta)$ and $\nu'_{i,k}(\theta)$ at the knots.

<table>
<thead>
<tr>
<th>$\theta^k_i$</th>
<th>$\theta^k_{i+1}$</th>
<th>$\theta^k_{i+2}$</th>
<th>$\theta^k_{i+3}$</th>
<th>else</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_{i,k}(\theta)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\nu'_{i,k}(\theta)$</td>
<td>0</td>
<td>$\frac{1}{2\tanh(\frac{h_k}{2})}$</td>
<td>$\frac{1}{2\tanh(\frac{h_k}{2})}$</td>
<td>0</td>
</tr>
</tbody>
</table>

3 Hyperbolic B-splines interpolation

In this section, we will construct an approximate of $y^{(4)}(\theta^k_j)$ by using Taylor series expansion. According to Schoenberg-Whitney theorem (see [5]), for a given function $y(\theta)$ sufficiently smooth there exists a unique hyperbolic spline $s(\theta) = \sum_{i=-2}^{m_k-3} \mu_i \nu_{i,k}(\theta) \in V_k$ satisfying the interpolation conditions:

\begin{align*}
s(\theta^k_j) &= y(\theta^k_j), \quad j = 0, 1, \ldots, m_k - 2; \\
s'(a) &= y'(a), \quad s'(b) = y'(b).
\end{align*}

For $j = 0, 1, \ldots, m_k - 3$, let $m_j = s'(\theta^k_j)$ and for $j = 1, 2, \ldots, m_k - 3$, let $M_j = \frac{s(\theta^k_j + h_k) - 2s(\theta^k_j) + s(\theta^k_j - h_k)}{h_k^2}$. By using the Taylor series expansion we have:

\begin{align*}
m_j &= s'(\theta^k_j) = y'(\theta^k_j) - \frac{1}{180} h_k^4 y^{(5)}(\theta^k_j) + O(h_k^6); \\
M_j &= y''(\theta^k_j) + \frac{1}{12} h_k^2 y^{(4)}(\theta^k_j) + \frac{1}{360} h_k^4 y^{(6)}(\theta^k_j) + O(h_k^6);
\end{align*}

Now we can applied $M_j$ to construct $y^{(3)}(\theta^k_j)$ and $y^{(4)}(\theta^k_j)$ for $j = 2, 3, \ldots, m_k - 4$, as follows, where the errors obtained by Taylor series expansion.

\begin{align*}
\frac{M_{j+1} - M_{j-1}}{2h_k} &= y^{(3)}(\theta^k_j) + \frac{1}{4} h_k^4 y^{(5)}(\theta^k_j) + O(h_k^6); \\
\frac{M_{j+1} - 2M_j + M_{j-1}}{h_k^2} &= y^{(4)}(\theta^k_j) + \frac{1}{6} h_k^4 y^{(6)}(\theta^k_j) + O(h_k^6);
\end{align*}

By Table 1 and this equations, we get:
Table 2: Approximation values of $y(\theta^k_j)$, $y'(\theta^k_j)$ and $y''(\theta^k_j)$.

<table>
<thead>
<tr>
<th>Approximate value</th>
<th>$y(\theta^k_j)$</th>
<th>$y'(\theta^k_j)$</th>
<th>$y''(\theta^k_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representation in $\mu_j$</td>
<td>$\mu_{j-2}+\mu_{j-1}$</td>
<td>$\mu_{j-1}+\mu_{j-2}$</td>
<td>$\mu_{j-3}+\mu_{j-2}$</td>
</tr>
<tr>
<td>Error order</td>
<td>$O(h^4_k)$</td>
<td>$O(h^4_k)$</td>
<td>$O(h^4_k)$</td>
</tr>
</tbody>
</table>

Table 3: Approximation values of $y^{(3)}(\theta^k_j)$ and $y^{(4)}(\theta^k_j)$.

<table>
<thead>
<tr>
<th>Approximate value</th>
<th>$y^{(3)}(\theta^k_j)$</th>
<th>$y^{(4)}(\theta^k_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representation in $\mu_j$</td>
<td>$\frac{\mu_{j-1}+\mu_{j-2}}{2h_k}$</td>
<td>$\frac{\mu_{j-3}+\mu_{j-2}+\mu_{j-1}+\mu_{j+1}}{2h_k}$</td>
</tr>
<tr>
<td>Error order</td>
<td>$O(h^2_k)$</td>
<td>$O(h^2_k)$</td>
</tr>
</tbody>
</table>

4 Hyperbolic B-splines solutions of 4BVP

Let $s(\theta) = \sum_{i=2}^{m_k-3} \mu_i \nu_{i,k}(\theta)$ be the approximate of (1) and $\tilde{s}(\theta) = \sum_{i=2}^{m_k-3} \tilde{\mu}_i \tilde{\nu}_{i,k}(\theta)$ be the approximate spline of $s(\theta)$. Discretize (1) at $\theta^k_j$ for $j = 2, \cdots, m_k - 4$, we get:

$$y^{(4)}(\theta^k_j) + f(\theta^k_j)y(\theta^k_j) = g(\theta^k_j), \quad j = 2, 3, \cdots, m_k - 4. \quad (10)$$

Now, by using Table 2 and 3, the equation (10) becomes

$$\mu_{j-4} - 3\mu_{j-3} + 2\mu_{j-2} + 2\mu_{j-1} - 3\mu_{j} + \mu_{j+1} + f_j \frac{\mu_{j-2} + \mu_{j-1}}{2} = g_j + O(h^4_k) \quad (11)$$

where $f_j = f(\theta^k_j)$ and $g_j = g(\theta^k_j)$. Consequently,

$$(\mu_{j-4} - 3\mu_{j-3} + 2\mu_{j-2} + 2\mu_{j-1} - 3\mu_{j} + \mu_{j+1}) + f_j h_k^4 (\mu_{j-2} + \mu_{j-1}) = 2g_j h_k^4 + O(h^4_k) \quad (12)$$

By dropping $O(h^8_k)$ from (12), we obtain a linear system with $m_k - 5$ linear equations in $m_k$ unknowns $\mu_j$, $j = 2, 3, \cdots, m_k - 4$. So five more equations are needed.

On the other hand, by using the fourth boundary conditions (2), we get

$$\begin{cases} 
\mu_{-2} = a_0; \\
\mu_{-1} - \mu_{-2} = 2a_1 \tanh(\frac{h_k}{2}) \end{cases} \quad \text{Thus,} \quad \begin{cases} 
\mu_{-2} = a_0; \\
\mu_{-1} = a_0 - 2a_1 \tanh(\frac{h_k}{2}) \end{cases} \quad (13)$$
By using a similar technique, we get:

\[
\begin{align*}
\mu_{m_k-3} &= b_0; \\
\mu_{m_k-3} - \mu_{m_k-4} &= 2b_1 \tanh\left(\frac{h_k}{2}\right) .
\end{align*}
\]

Thus,

\[
\begin{align*}
\mu_{m_k-3} &= b_0 \\
\mu_{m_k-4} &= b_0 + 2b_1 \tanh\left(\frac{h_k}{2}\right)
\end{align*}
\]  

(14)

For the fifth equation we propose the following formula

\[
y_j^{(1)}(\theta) + 4y_j^{(2)}(\theta) + y_j^{(3)}(\theta) = \frac{3}{h_k} (y_{j+1}(\theta) - y_{j-1}(\theta)) + O(h_k^4)
\]

(15)

which can be easily demonstrated using a Taylor series expansion. By formulae (15), Table 2 and 3, we can construct an approximate formulae for \( y^{(4)}(\alpha) \), as follows

\[
y^{(4)}(\alpha) = \frac{2M_3 - 6M_2 + 8M_1 - \Omega_{h_k}}{h_k^2} + O(h_k^4)
\]

(16)

where \( \Omega_{h_k} = \frac{3}{h_k^2}(m_2 - m_0) \). For smaller \( h_k \ (\tanh(h_k/2) \sim h_k^4) \) and by turning (16), the coefficients are determined as follows

\[
3\mu_0 + 3\mu_1 - 4\mu_2 + \mu_3 = h_k^4 g_0 - h_k^4 f_0 a_0 - \mu_2 - 4\mu_1 + O(h_k^8)
\]

(17)

Take (12) and (17), we get \( m_k - 4 \) linear equations with \( \mu_i \), \( i = 0, 1, \ldots, m_k - 5 \), as unknowns since \( \mu_{-2}, \mu_{-1}, \mu_{m_k-4} \) and \( \mu_{m_k-3} \) have been yielded from (13) and (14).

Let \( C = [\mu_0, \mu_1, \ldots, \mu_{m_k-5}]^T \), \( \tilde{C} = [\tilde{\mu}_0, \tilde{\mu}_1, \ldots, \tilde{\mu}_{m_k-5}]^T \), \( D = [d_0, d_1, \ldots, d_{m_k-5}]^T \), \( E = [e_0, e_1, \ldots, e_{m_k-5}]^T \) and using equations (12), (13), (14) and (17), we get

\[
(A + h_k^4 FB)C = D + E; \quad (A + h_k^4 FB)\tilde{C} = D,
\]

(18)

(19)

where \( A \) and \( B \) are the following \( (m_k - 4) \times (m_k - 4) \) matrix:

\[
A = \begin{pmatrix}
3 & 3 & -4 & 1 \\
2 & 2 & -3 & 1 \\
-3 & 2 & 2 & -3 & 1 \\
1 & -3 & 2 & 2 & -3 & 1 \\
& & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & -3 & 2 & 2 & -3 & 1 \\
1 & -3 & 2 & 2 & -3 & 1 \\
1 & -3 & 2 & 2 & 2 & \end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 \\
1 & 1 \\
1 & 1 \\
& & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

and where \( F \) and \( D \) are the following matrix
On the other hand, using the fact that $z \mu \beta$ we deduce $z$ and $e_k$ such that the interval $\left[ \frac{4}{7} \right]$ get the approximation spline solution $\tilde{s}(\theta) = \sum_{i=-2}^{m_{k-3}} \tilde{\mu}_i \nu_{i,k}(\theta)$.

In order to determine the bound of $\| C - \tilde{C} \|_\infty$, we need the following lemma.

**Lemma 4.1** The matrix $A$ is inversible.

**PROOF.** It suffices to prove that for all $D = [d_0, d_1, \cdots, d_{m_k-5}]^T \in \mathbb{R}^{m_k-4}$ such that $A D = 0$, we have $D = 0$. Indeed, If we put $z(\theta) = \sum_{i=-2}^{m_{k-3}} d_{j+2} \nu_{j,k}(\theta) + \sum_{j=m_k-6}^{m_k-3} 0 \nu_{j,k}(\theta)$, then $z^{(4)}(\theta^k_i) = 0$, for all $i = 4, 5, \cdots, m_k - 6$.

On the other hand, using the fact that $z$ is hyperbolic spline function of $C^1$, we deduce that $z(\theta) = \alpha + \beta \cosh(\theta) + \gamma \sinh(\theta)$ in $[\theta^k_4, \theta^k_5]$. From $z^{(4)}(\theta^k_4) = 0$ and $z^{(4)}(\theta^k_5) = 0$, we have,

$$\left\{ \begin{array}{l}
\beta \cosh(\theta^k_4) + \gamma \sinh(\theta^k_4) = 0; \\
\beta \cosh(\theta^k_5) + \gamma \sinh(\theta^k_5) = 0;
\end{array} \right.$$  

we deduce $\beta = 0$ and $\gamma = 0$. Consequently, $z^{(4)}(\theta) = 0$ and $z'(\theta) = 0$ in all the interval $[\theta^k_4, \theta^k_5]$. In a same way, we have in all the other subintervals of $[\theta^k_4, \theta^k_{m_k-6}]$, $z^{(4)}(\theta) = 0$ and $z'(\theta) = 0$. Consequently, we have

$$\left\{ \begin{array}{l}
z'(\theta^k_4) = 0 \\
z'(\theta^k_5) = 0 \\
\vdots \\
z'(\theta^k_{m_k-7}) = 0 \\
z'(\theta^k_{m_k-6}) = 0
\end{array} \right.$$  

thus,

$$\left\{ \begin{array}{l}
\frac{d_5 - d_4}{2 \tanh(\frac{\theta^k_4}{2})} = 0 \\
\frac{d_6 - d_5}{2 \tanh(\frac{\theta^k_5}{2})} = 0 \\
\vdots \\
\frac{d_{m_k-6} - d_{m_k-7}}{2 \tanh(\frac{\theta^k_{m_k-6}}{2})} = 0
\end{array} \right.$$
so \( d_4 = d_5 = d_6 = \cdots = d_{m_k-6} = d_{m_k-5} \), we have also

\[
\begin{pmatrix}
3d_0 + 3d_1 - 4d_2 + d_3 = 0 \\
2d_0 + 2d_1 - 3d_2 + d_3 = 0 \\
-3d_0 + 2d_1 + 2d_2 - 3d_3 + d_4 = 0 \\
d_0 - 3d_1 + 2d_2 + 2d_3 - 3d_4 + d_5 = 0 \\
& \vdots \\
d_{m_k-9} - 3d_{m_k-8} + 2d_{m_k-7} + 2d_{m_k-6} - 3d_{m_k-5} = 0 \\
d_{m_k-8} - 3d_{m_k-7} + 2d_{m_k-6} + 2d_{m_k-5} = 0
\end{pmatrix} 
\]

(20)

finally we have \( d_0 = d_1 = d_2 = \ldots = d_{m_k-5} = 0 \) which in turn gives \( D = 0 \).

**Lemma 4.2** If we assume that \( h_k^4 \| A^{-1} \|_\infty \| B \|_\infty \| F \|_\infty \leq \frac{1}{2} \), then there exists a constant \( K \) which depends only of the functions \( f \) and \( g \) such that

\[
\| C - \tilde{C} \|_\infty \leq Kh_k.
\]

**Proof.** From (18), (19) and Lemma 1, we have \( C - \tilde{C} = (I + h_k^4 A^{-1} BF)^{-1} A^{-1} E \). Since \( E = O(h_k^8) \), there exists a constant \( K_1 \) such that \( \| E \|_\infty \leq K_1 h_k^8 \). Consequently

\[
\| C - \tilde{C} \|_\infty \leq K_1 h_k^8 \| A^{-1} \|_\infty \| (I + h_k^4 A^{-1} BF)^{-1} \|_\infty.
\]

Using the inequality \( h_k^4 \| A^{-1} \|_\infty \| B \|_\infty \| F \|_\infty \leq \frac{1}{2} \), and \( \| B \|_\infty = 2 \), we deduce that

\[
\| C - \tilde{C} \|_\infty \leq K_1 \frac{h_k^4 \| A^{-1} \|_\infty}{1 - h_k^4 \| A^{-1} \|_\infty \| B \|_\infty \| F \|_\infty} h_k \leq \frac{K_1 (b - a)^3}{\| 2F \|_\infty} h_k.
\]

Hence, \( |s(\theta) - \widetilde{s}(\theta)| \leq \| C - \tilde{C} \|_\infty \sum_{i=-2}^{m_k-3} \nu_{i,k}(\theta) = \| C - \tilde{C} \|_\infty \approx O(h_k) \). Therefore, we get

\[
|y(\theta) - \widetilde{s}(\theta)| \leq |y(\theta) - s(\theta)| + |s(\theta) - \widetilde{s}(\theta)| \leq O(h_k^4) + O(h_k) \approx O(h_k).
\]

**5 Numerical results**

To test our method, we considered two examples of fourth-order boundary value problems (4BVPs) of the form \((2),(1))\).

**EXAMPLE 1:** We consider the following boundary-value problem

\[
\begin{cases}
y^{(4)}(x) + xy(x) = -(8 + 7x + x^3)e^x, & x \in [0, 1]; \\
y(0) = y(1) = 0, \quad y'(0) = 1, \quad y'(1) = -e.
\end{cases}
\]

(21)
Table 4: Maximum absolute error and order of convergence for Problem (21).

<table>
<thead>
<tr>
<th>$k$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>3.129e-002</td>
<td>1.508e-002</td>
<td>7.409e-003</td>
<td>3.672e-003</td>
<td>1.828e-003</td>
<td>9.120e-004</td>
</tr>
<tr>
<td>Order</td>
<td>1.0531</td>
<td>1.0253</td>
<td>1.0127</td>
<td>1.0063</td>
<td>1.0032</td>
<td></td>
</tr>
</tbody>
</table>

The exact solution is $y(x) = x(1 - x)e^x$.

Results have been shown for different values of $k$ in Table 4.

EXAMPLE 2: We consider the following boundary-value problem

\[
\begin{cases}
  y^{(4)}(x) - y(x) = 4((1 - 2x)\sinh(x) - 3\cosh(x)), & x \in [0, 1]; \\
  y(0) = y(1) = 0, \ y'(0) = 1, \ y'(1) = -\cosh(x).
\end{cases}
\]  

(22)

The exact solution is $y(x) = x(1 - x)\cosh(x)$.

Results have been shown for different values of $k$ in Table 5.

Table 5: Maximum absolute error and order of convergence for Problem (22).

<table>
<thead>
<tr>
<th>$k$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>2.140e-002</td>
<td>1.031e-002</td>
<td>5.067e-003</td>
<td>2.511e-003</td>
<td>1.250e-003</td>
<td>6.238e-004</td>
</tr>
<tr>
<td>Order</td>
<td>1.0536</td>
<td>1.0248</td>
<td>1.0129</td>
<td>1.0063</td>
<td>1.0028</td>
<td></td>
</tr>
</tbody>
</table>

6 Conclusion

Numerical results confirm the order of convergence predicted by the analysis. Experimental results demonstrate that our method is more effective for the problems where the exact solution is hyperbolic (see Table 5). The proposed method can be extended to solve higher (i.e., more than 5th) order boundary value problems by using the hyperbolic (tension) B-splines of order more than 4, 5, ⋯.

References


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