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Poly-Cauchy Numbers and Polynomials with Umbral Calculus Viewpoint

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Abstract. In this paper, we give some interesting identities of poly-Cauchy numbers and polynomials arising from umbral calculus.

1. Introduction

For $k \in \mathbb{Z}$, the polylogarithm factorial function is defined by

$$Lif_k(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!(m+1)^k}, \ (see \ [11, 12, 13]). \qquad (1)$$

When $k=1$, $Lif_1(t)=\sum_{m=0}^{\infty}\frac{t^m}{(m+1)!}=\frac{e^t-1}{t}$.

The poly-Cauchy polynomials of the first kind $C_n^{(k)}(x)$ of index k are defined by the generating function to be

$$\frac{Lif_k(log(1+t))}{(1+t)^x}=\sum_{n=0}^{\infty}C_n^{(k)}(x)\frac{t^n}{n!},\ (k\in\mathbb{Z}),\ (see[11]). \tag{2}$$

For $\lambda\in\mathbb{C}$ with $\lambda\neq 1$, the n-th Frobenius-Euler polynomial of order r is defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|\lambda)\frac{t^n}{n!},\ (r\in\mathbb{Z}),\ (\text{see }[1,10]). \tag{3}$$

As is well known, the n-th Bernoulli polynomial of order r is given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\ (r\in\mathbb{Z}),\ (\text{see }[1,15]). \tag{4}$$

The Stirling numbers of the first kind are defined by the generating function to be

$$(\log(1+t))^m=m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}\ (m\in\mathbb{Z}_{\geq 0}). \tag{5}$$

Thus, by (5), we get

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l, \tag{6}$$

and

$$x^{(n)}=x(x+1)\cdots(x+n-1)=\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)x^l,\ (see\ [14]). \tag{7}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{ f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\ \middle|\ a_k\in\mathbb{C}\right\}. \tag{8}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L \mid p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$.

For $f(t) \in \mathcal{F}$ with $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \ (n \geq 0), \ (\text{see [2,14,15]}). \tag{9}$$

By (8) and (9), we easily see that

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k}, \ (n, k \geq 0), \tag{10}$$

where $\delta_{n,k}$ is the Kronecker symbol. (see [14,15]).

Let us consider $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^n\rangle}{k!} t^k$. By (10), we easily see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$ and so as linear functionals $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional (see[14,15]). We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [14]). The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series; if $o(g(t)) = 0$, then $g(t)$ is called an invertible seires. Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ ($\deg S_n(x) = n$) such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle = n!\delta_{n,k}$ $for\ n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$, $(see$ [14]).

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle, \tag{11}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \ p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!}. \tag{12}$$

By (12), we get

$$p^{(k)}(0) = \left\langle t^k|p(x)\right\rangle = \left\langle 1|p^{(k)}(x)\right\rangle \ (k \geq 0). \tag{13}$$

Thus, by (13), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \ (see\ [14]). \tag{14}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we see that

$$\frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}, \ for \ all \ x \in \mathbb{C}, \tag{15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$,

$$S_n(x) = \sum_{j=0}^{n} \frac{1}{j!}\langle \frac{\bar{f}(t)^j}{g(\bar{f}(t))}|x^n\rangle x^j, \tag{16}$$

and

$$f(t)S_n(x) = nS_{n-1}(x), \ (n \geq 0), \ (see \ [14, 15]). \tag{17}$$

As is well known, the transfer formula for $p_n(x) \sim (1, f(t))$, $q_n \sim (1, g(t))$, $(n \geq 1)$, is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x), \ (see \ [14]). \tag{18}$$

Let $S_n(x) \sim (g(t), f(t))$. Then it is known that

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}S_n(x), \ (see \ [14]). \tag{19}$$

For $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$ we have

$$S_n(x) = \sum_{m=0}^{n} C_{n,m} r_n(x), \tag{20}$$

where

$$C_{n,m} = \frac{1}{m!}\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m|x^n\rangle, \ (see \ [14]). \tag{21}$$

Finally, we note that $e^{yt}p(x) = p(x+y)$, $(p(x) \in \mathbb{P})$.

In this paper, we investigate some properties of poly-Cauchy numbers and polynomials with umbral calculus viewpoint. From our investigation, we derive some interesting identities of poly-Cauchy numbers and polynomials.

2. Poly-Cauchy numbers and polynomials

From (2), we note that $C_n^{(k)}(x)$ is the Sheffer sequence for the pair $(g(t) = \frac{1}{Lif_k(-t)}, \ f(t) = e^{-t} - 1)$, that is,

$$C_n^{(k)}(x) \sim \left(\frac{1}{Lif_k(-t)}, e^{-t} - 1\right), \ (k \in \mathbb{Z}, \ n \geq 0). \tag{22}$$

When $x = 0$, $C_n^{(k)} = C_n^{(k)}(0)$ is called the n-th poly-Cauchy number of the first kind with index k.

Thus, we see that

$$Lif_k(log(1+t)) = \sum_{m=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}, \ (see\ [12, 13]). \tag{23}$$

By (16) and (22), we easily get

$$C_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle Lif_k(log(1+t))(-log(1+t))^j | x^n \rangle x^j. \tag{24}$$

Now, we compute.

$$\begin{aligned} \langle Lif_k(log(1+t))(-log(1+t))^j | x^n \rangle &= (-1)^j \sum_{m=0}^{\infty} \frac{1}{m!(m+1)^k} \langle (log(1+t))^{m+j} | x^n \rangle \\ &= (-1)^j \sum_{m=0}^{n-j} \frac{1}{m!(m+1)^k} \sum_{l=0}^{n-j-m} \frac{(m+j)!}{(l+m+j)!} S_1(l+m+j, m+j)(l+m+j)! \delta_{n,l+m+j} \\ &= (-1)^j \sum_{m=0}^{n-j} \frac{(m+j)!}{m!(m+1)^k} S_1(n, m+j). \end{aligned} \tag{25}$$

Thus, by (24) and (25), we get

$$\begin{aligned} C_n^{(k)}(x) &= \sum_{j=0}^{n} \{(-1)^j \sum_{m=0}^{n-j} \frac{\binom{m+j}{m}}{(m+1)^k} S_1(n, m+j)\} x^j \\ &= \sum_{j=0}^{n} \{(-1)^j \sum_{m=j}^{n} \frac{\binom{m}{j}}{(m-j+1)^k} S_1(n, m)\} x^j \\ &= \sum_{m=0}^{n} S_1(n, m) \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(m-j+1)^k}. \end{aligned} \tag{26}$$

From (22), we have

$$\frac{1}{Lif_k(-t)} C_n^{(k)}(x) \sim (1, e^{-t} - 1), \ x^n \sim (1, t). \tag{27}$$

By (18) and (22), for $n \geq 1$ we get

$$\frac{1}{Lif_k(-t)} C_n^{(k)}(x) = x \left(\frac{t}{e^{-t} - 1} \right)^n x^{-1} x^n = (-1)^n x \left(\frac{te^t}{e^t - 1} \right)^n x^{n-1}. \tag{28}$$

Now, we observe that

$$\left(\frac{te^t}{e^t-1}\right)^n = \left(\frac{-t}{e^{-t}-1}\right)^n = \left(\sum_{l_1=0}^{\infty}\frac{(-1)^{l_1}t^{l_1}}{l_1!}B_{l_1}\right)\times\cdots\times\left(\sum_{l_n=0}^{\infty}\frac{(-1)^{l_n}B_{l_n}}{l_n!}t^{l_n}\right)$$
$$= \sum_{l=0}^{\infty}\left(\sum_{l_1+\cdots+l_n=l}(-1)^l\binom{l}{l_1,\cdots,l_n}B_{l_1}\cdots B_{l_n}\right)\frac{t^l}{l!}, \tag{29}$$

where B_n is the n-th ordinary Bernoulli number and $\binom{n}{l_1,\cdots,l_n} = \frac{n!}{l_1!l_2!\cdots l_n!}$. From (28) and (29), we have

$$\begin{aligned}
&C_n^{(k)}(x)\\
&= (-1)^n\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}(-1)^l\binom{n-1}{l_1,\cdots,l_n,n-1-l}B_{l_1}\cdots B_{l_n}Lif_k(-t)x^{n-l}\\
&= (-1)^n\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\sum_{m=0}^{n-l}(-1)^{m+l}\binom{n-1}{l_1,\cdots,l_n,n-1-l}\binom{n-l}{m}\\
&\qquad\qquad \times\frac{1}{(m+1)^k}B_{l_1}\cdots B_{l_n}x^{n-l-m}\\
&= (-1)^n\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\sum_{j=0}^{n-l}(-1)^{n-j}\binom{n-1}{l_1,\cdots,l_n,n-1-l}\binom{n-l}{j}\frac{B_{l_1}\cdots B_{l_n}}{(n-l-j+1)^k}x^j\\
&= \sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\binom{n-1}{l_1,\cdots,l_n,n-1-l}\frac{B_{l_1}\cdots B_{l_n}}{(n-l+1)^k}\\
&\quad + \sum_{j=1}^{n}\left\{\sum_{l=0}^{n-j}\sum_{l_1+\cdots+l_n=l}(-1)^j\binom{n-1}{l_1,\cdots,l_n,n-1-l}\binom{n-l}{j}\frac{B_{l_1}\cdots B_{l_n}}{(n-l-j+1)^k}\right\}x^j.
\end{aligned} \tag{30}$$

Therefore, by (30), we obtain the following theorem

Theorem 2.1. *For $k\in\mathbb{Z}$, $n\geq 1$, we have*

$$\begin{aligned}
C_n^{(k)}(x) &= \sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\binom{n-1}{l_1,\cdots,l_n,n-1-l}\frac{B_{l_1}\cdots B_{l_n}}{(n-l+1)^k}\\
&\quad + \sum_{j=1}^{n}\left\{\sum_{l=0}^{n-j}\sum_{l_1+\cdots+l_n=l}(-1)^j\binom{n-1}{l_1,\cdots,l_n,n-1-l}\binom{n-l}{j}\frac{B_{l_1}\cdots B_{l_n}}{(n-l-j+1)^k}\right\}x^j.
\end{aligned}$$

From (28), we have

$$\begin{aligned}
\frac{1}{Lif_k(-t)}C_n^{(k)}(x) &= (-1)^n x\left(\frac{te^t}{e^t-1}\right)^n x^{n-1} = (-1)^n x\left(t+\frac{t}{e^t-1}\right)^n x^{n-1} \\
&= (-1)^n x\sum_{a=0}^{n}\binom{n}{a}t^{n-a}\left(\frac{t}{e^t-1}\right)^a x^{n-1} \\
&= (-1)^n x\sum_{a=0}^{n}\binom{n}{a}t^{n-a}B_{n-1}^{(a)}(x) \\
&= (-1)^n \sum_{a=1}^{n}\binom{n}{a}(n-1)_{n-a}xB_{a-1}^{(a)}(x) \\
&= n!(-1)^n \sum_{a=1}^{n}\frac{1}{a!}\binom{n-1}{a-1}xB_{a-1}^{(a)}(x).
\end{aligned} \tag{31}$$

Thus, by (31), we get

$$\begin{aligned}
C_n^{(k)}(x) &= (-1)^n n!\sum_{a=1}^{n}\sum_{l=0}^{a-1}\frac{1}{a!}\binom{n-1}{a-1}\binom{a-1}{l}B_{a-1-l}^{(a)}Lif_k(-t)x^{l+1} \\
&= (-1)^n n!\sum_{a=1}^{n}\sum_{l=0}^{a-1}\sum_{m=0}^{l+1}\frac{1}{a!}\binom{n-1}{a-1}\binom{a-1}{l}\binom{l+1}{m}\frac{(-1)^{l+1-m}}{(l+2-m)^k}B_{a-1-l}^{(a)}x^m \\
&= (-1)^n n!\left\{\sum_{a=1}^{n}\sum_{l=0}^{a-1}\frac{(-1)^{l+1}}{a!}\binom{n-1}{a-1}\binom{a-1}{l}\frac{B_{a-1-l}^{(a)}}{(l+2)^k}\right. \\
&\left.+\sum_{a=1}^{n}\sum_{l=0}^{a-1}\sum_{m=1}^{l+1}\frac{1}{a!}\binom{n-1}{a-1}\binom{a-1}{l}\binom{l+1}{m}\frac{(-1)^{l+1-m}}{(l+2-m)^k}B_{a-1-l}^{(a)}x^m\right\} \\
&= (-1)^n n!\left\{\sum_{a=1}^{n}\sum_{l=0}^{a-1}\frac{(-1)^{l+1}}{a!}\binom{n-1}{a-1}\binom{a-1}{l}\frac{B_{a-1-l}^{(a)}}{(l+2)^k}\right. \\
&\left.+\sum_{m=1}^{n}\sum_{a=m}^{n}\sum_{l=m-1}^{a-1}\frac{(-1)^{l+1-m}}{a!}\binom{n-1}{a-1}\binom{a-1}{l}\binom{l+1}{m}\frac{B_{a-1-l}^{(a)}}{(l+2-m)^k}x^m\right\}.
\end{aligned} \tag{32}$$

Therefore, by (26),(30) and (32), we obtain the following theorem.

Theorem 2.2. *For $n \geq 1,\ 1 \leq j \leq n$, we have*

$$\begin{aligned}
&\sum_{m=j}^{n} \frac{\binom{m}{j}}{(m-j+1)^k} S_1(n,m) \\
&= \sum_{l=0}^{n-j} \sum_{l_1+\cdots+l_n=l} \binom{n-1}{l_1,\cdots,l_n,n-1-l} \binom{n-l}{j} \frac{B_{l_1}\cdots B_{l_n}}{(n-l-j+1)^k} \\
&= (-1)^n n! \sum_{a=j}^{n} \sum_{l=j-1}^{a-1} \frac{(-1)^{l+1-j}}{a!} \binom{n-1}{a-1} \binom{a-1}{l} \binom{l+1}{j} \frac{B_{a-1-l}^{(a)}}{(l+2-j)^k}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
C_n^{(k)} &= \sum_{m=0}^{n} \frac{S_1(n,m)}{(m+1)^k} = \sum_{l=0}^{n-1} \sum_{l_1+\cdots+l_n=l} \binom{n-1}{l_1,\cdots,l_n,n-1-l} \frac{B_{l_1}\cdots B_{l_n}}{(n-l+1)^k} \\
&= (-1)^n n! \sum_{a=1}^{n} \sum_{l=0}^{a-1} \frac{(-1)^{l+1}}{a!} \binom{n-1}{a-1} \binom{a-1}{l} \frac{B_{a-1-l}^{(a)}}{(l+2)^k}.
\end{aligned}$$

where $k \in \mathbb{Z},\ n \geq 1$.

From (7), we note that

$$\frac{1}{(1-t)^x} = \sum_{n=0}^{\infty} (-x)_n \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} x^{(n)} \frac{t^n}{n!}. \tag{33}$$

By (15) and (33), we get

$$x^{(n)} = \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) x^m \sim (1, 1-e^{-t}), \tag{34}$$

and

$$(-1)^n x^{(n)} = \sum_{m=0}^{n} (-1)^m S_1(n,m) x^m \sim (1, e^{-t}-1). \tag{35}$$

Thus, by (27) and (35), we get

$$\frac{1}{Lif_k(-t)} C_n^{(k)}(x) = (-1)^n x^{(n)} \Leftrightarrow C_n^{(k)}(x) = (-1)^n Lif_k(-t) x^{(n)}. \tag{36}$$

From (36), we have

$$
\begin{aligned}
C_n^{(k)}(x) &= Lif_k(-t)(-1)^n x^{(n)} = \sum_{m=0}^{n}(-1)^m S_1(n,m) Lif_k(-t)x^m \\
&= \sum_{m=0}^{n}(-1)^m S_1(n,m)\sum_{a=0}^{\infty}\frac{(-1)^a}{a!(a+1)^k}t^a x^m \\
&= \sum_{m=0}^{n}(-1)^m S_1(n,m)\sum_{a=0}^{m}\frac{(-1)^a}{a!(a+1)^k}(m)_a x^{m-a} \\
&= \sum_{m=0}^{n}\sum_{j=0}^{m}(-1)^m S_1(n,m)\frac{(-1)^{m-j}}{(m-j)!(m-j+1)^k}(m)_{m-j}x^j \\
&= \sum_{m=0}^{n}\sum_{j=0}^{m} S_1(n,m)\binom{m}{j}\frac{(-x)^j}{(m-j+1)^k}.
\end{aligned}
\tag{37}
$$

It is well known that the Sheffer identity is given by

$$
S_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}S_j(x)P_{n-j}(y),\ (see\ [14]), \tag{38}
$$

where $S_n(x) \sim (g(t), f(t))$ and $P_n(x) = g(t)S_n(x)$.

By (38), we easily get

$$
C_n^{(k)}(x+y) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}C_j^{(k)}(x)y^{(n-j)}, \tag{39}
$$

where $y^{(n)} = y(y+1)\cdots(y+n-1)$.

From (19) and (22), we have

$$
\begin{aligned}
C_{n+1}^{(k)}(x) &= \left(x - \frac{Lif_k'(-t)}{Lif_k(-t)}\right)(-e^t)C_n^{(k)}(x) \\
&= e^t\frac{Lif_k'(-t)}{Lif_k(-t)}C_n^{(k)}(x) - xC_n^{(k)}(x+1).
\end{aligned}
\tag{40}
$$

Now, we compute

$$
\begin{aligned}
\frac{Lif_k'(-t)}{Lif_k(-t)}C_n^{(k)}(x) &= Lif_k'(-t)\left(\frac{1}{Lif_k(-t)}C_n^{(k)}(x)\right) \\
&= Lif_k'(-t)(-1)^n x^{(n)} = (-1)^n Lif_k'(-t)x^{(n)} \\
&= (-1)^n\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)Lif_k'(-t)x^l.
\end{aligned}
\tag{41}
$$

By the definition of the polylogarithm factorial function, we get

$$Lif'_k(-t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!(n+2)^k}. \tag{42}$$

From (41) and (42), we can derive

$$\begin{aligned}\frac{Lif'_k(-t)}{Lif_k(-t)}C_n^{(k)}(x)&=(-1)^n\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)\sum_{m=0}^{l}(-1)^m\binom{l}{m}\frac{x^{l-m}}{(m+2)^k}\\&=\sum_{l=0}^{n}S_1(n,l)\sum_{j=0}^{l}\frac{(-1)^j\binom{l}{j}}{(l-j+2)^k}x^j.\end{aligned}\tag{43}$$

Thus, by (40) and (43), we get

$$\begin{aligned}C_{n+1}^{(k)}(x)&=e^t\sum_{l=0}^{n}S_1(n,l)\sum_{j=0}^{l}\frac{(-1)^j\binom{l}{j}}{(l-j+2)^k}x^j-xC_n^{(k)}(x+1)\\&=\sum_{l=0}^{n}S_1(n,l)\sum_{j=0}^{l}\frac{(-1)^j\binom{l}{j}}{(l-j+2)^k}(x+1)^j-xC_n^{(k)}(x+1).\end{aligned}\tag{44}$$

Therefore, we obtain the following theorem.

Theorem 2.3. *For $k\in\mathbb{Z}$, $n\geq 0$, we have*

$$C_{n+1}^{(k)}(x)=\sum_{l=0}^{n}S_1(n,l)\sum_{j=0}^{l}\frac{(-1)^j\binom{l}{j}}{(l-j+2)^k}(x+1)^j-xC_n^{(k)}(x+1).$$

For $f(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$, we note that

$$\langle f(t)|xp(x)\rangle=\langle\partial_t f(t)|p(x)\rangle,\ (see\ [14]), \tag{45}$$

where $\partial_t f(t)=\frac{df(t)}{dt}$.

By (10) and (45), we get

$$\begin{aligned}C_n^{(k)}(y)&=\langle\sum_{l=0}^{\infty}C_l^{(k)}(y)\frac{t^l}{l!}|x^n\rangle=\langle\frac{Lif_k(\log(1+t))}{(1+t)^y}|xx^{n-1}\rangle\\&=\langle\partial_t(Lif_k(\log(1+t))(1+t)^{-y})|x^{n-1}\rangle\\&=\langle(\partial_t Lif_k(\log(1+t)))(1+t)^{-y}|x^{n-1}\rangle+\langle Lif_k(\log(1+t))\partial_t(1+t)^{-y}|x^{n-1}\rangle\\&=\langle Lif'_k(\log(1+t))(1+t)^{-y-1}|x^{n-1}\rangle-yC_{n-1}^{(k)}(y+1).\end{aligned}\tag{46}$$

It is easy to show that

$$(tLif_k(t))' = Lif_{k-1}(t),\ \ (tLif_k(t))' = Lif_k(t) + tLif_k'(t). \tag{47}$$

Thus, by (47), we get

$$Lif_k'(t) = \frac{Lif_{k-1}(t) - Lif_k(t)}{t} \tag{48}$$

From (48), we can derive the following equation:

$$\begin{aligned}
&\langle Lif_k'(\log(1+t))(1+t)^{-y-1}|x^{n-1}\rangle \\
&= \langle \frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{t}(1+t)^{-y-1}|\frac{t}{\log(1+t)}x^{n-1}\rangle \\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(l)}(1)\langle\frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{t}(1+t)^{-y-1}|x^{n-1-l}\rangle \\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(l)}(1)\langle\frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{t}(1+t)^{-y-1}|\frac{1}{n-l}tx^{n-l}\rangle \\
&= \sum_{l=0}^{n-1}\binom{n-1}{l}\frac{B_l^{(l)}(1)}{n-l}\langle(Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t)))\,(1+t)^{-y-1}|x^{n-l}\rangle \\
&= \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}B_l^{(l)}(1)\{C_{n-l}^{(k-1)}(y+1) - C_{n-l}^{(k)}(y+1)\}.
\end{aligned} \tag{49}$$

Therefore, by (46) and (49), we obtain the following theorem.

Theorem 2.4. *For $k \in \mathbb{Z},\ n \geq 0$, we have*

$$C_n^{(k)}(x) = -xC_{n-1}^{(k)}(x+1) + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_l^{(l)}(1)\{C_{n-l}^{(k-1)}(x+1) - C_{n-l}^{(k)}(x+1)\}.$$

For $n \geq m \geq 1$, we evaluate

$$\langle(\log(1+t))^m Lif_k(\log(1+t))|x^n\rangle \tag{50}$$

in two different ways.

On the one hand, we get

$$\begin{aligned}
&\langle(\log(1+t))^m Lif_k(\log(1+t))|x^n\rangle = \langle Lif_k(\log(1+t))|(\log(1+t))^m x^n\rangle \\
&= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m}\langle Lif_k(\log(1+t))|x^{n-l-m}\rangle \\
&= \sum_{l=0}^{n-m} m!\binom{n}{l+m} S_1(l+m,m) C_{n-l-m}^{(k)}.
\end{aligned} \tag{51}$$

On the other hand, we have

$$\begin{aligned}
&\langle(\log(1+t))^m Lif_k(\log(1+t))|x^n\rangle = \langle(\log(1+t))^m Lif_k(\log(1+t))|xx^{n-1}\rangle \\
&= \langle\partial_t(\log(1+t))^m Lif_k(\log(1+t))|x^{n-1}\rangle.
\end{aligned} \tag{52}$$

Now, we observe that

$$\begin{aligned}
&\partial_t((\log(1+t))^m Lif_k(\log(1+t))) = \partial_t\{(\log(1+t))^{m-1}\log(1+t)Lif_k(\log(1+t))\} \\
&= (\partial_t(\log(1+t))^{m-1})\log(1+t)Lif_k(\log(1+t)) + (\log(1+t))^{m-1} \\
&\qquad \times(\partial_t(\log(1+t))Lif_k(\log(1+t))) \\
&= (\log(1+t))^{m-1}\frac{1}{1+t}\{(m-1)Lif_k(\log(1+t)) + Lif_{k-1}(\log(1+t))\}.
\end{aligned} \tag{53}$$

By (52) and (53), we get

$$\begin{aligned}
&\langle(\log(1+t))^m Lif_k(\log(1+t))|x^n\rangle \\
&= (m-1)\langle Lif_k(\log(1+t))(1+t)^{-1}|(\log(1+t))^{m-1}x^{n-1}\rangle \\
&\qquad + \langle(Lif_{k-1}(\log(1+t))(1+t)^{-1}|(\log(1+t))^{m-1}x^{n-1}\rangle \\
&= \sum_{l=0}^{n-m}(m-1)!\binom{n-1}{l+m-1}S_1(l+m-1,m-1)\{(m-1)C_{n-l-m}^{(k)}(1) + C_{n-l-m}^{(k-1)}(1)\}.
\end{aligned} \tag{54}$$

Therefore, by (51) and (54), we obtain the following theorem.

Theorem 2.5. *For $n \geq m \geq 1$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n-m} m!\binom{n}{l+m} S_1(l+m,m) C_{n-l-m}^{(k)} \\
&= \sum_{l=0}^{n-m}(m-1)!\binom{n-1}{l+m-1}S_1(l+m-1,m-1)\{(m-1)C_{n-l-m}^{(k)}(1) + C_{n-l-m}^{(k-1)}(1)\}.
\end{aligned}$$

In particular, for $n \geq 1$, we have

$$C_{n-1}^{(k-1)}(1) = \sum_{l=0}^{n-1} (-1)^l l! \binom{n}{l+1} C_{n-l-1}^{(k)}.$$

From (1), we note that

$$\sum_{n=0}^{\infty} C_n \frac{t^n}{n!} = Lif_1(\log(1+t)) = \frac{t}{\log(1+t)}, \tag{55}$$

where $C_n = C_n^{(1)}(0)$ is called the n-th Cauchy number of the first kind.

Let us consider the following sequences which are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t)) = \sum_{n=0}^{\infty} T_n^{(r,k)} \frac{t^n}{n!}. \tag{56}$$

Then, by (55) and (56), we get

$$\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t)) = \sum_{n=0}^{\infty} \left\{ \sum_{l_1+\cdots+l_{r+1}=n} \binom{n}{l_1, \cdots, l_{r+1}} C_{l_1} \cdots C_{l_r} C_{l_{r+1}}^{(k)} \right\} \frac{t^n}{n!}. \tag{57}$$

From (56) and (57), we have

$$T_n^{(r,k)} = \sum_{l_1+\cdots+l_{r+1}=n} \binom{n}{l_1, \cdots, l_{r+1}} C_{l_1} \cdots C_{l_r} C_{l_{r+1}}^{(k)}. \tag{58}$$

For $n \geq 1$, by (10), we get

$$\begin{aligned} C_n^{(k)} &= \langle Lif_k(\log(1+t)|x^n\rangle = \langle Lif_k(\log(1+t)|xx^{n-1}\rangle \\ &= \langle \partial_t(Lif_k(\log(1+t))|x^{n-1}\rangle = \langle \frac{Lif_{k-1}(\log(1+t) - Lif_k(\log(1+t)}{(1+t)\log(1+t)}|x^{n-1}\rangle \\ &= \langle \frac{Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t))}{(1+t)\log(1+t)}|\frac{1}{n}tx^n\rangle \\ &= \frac{1}{n}\langle \frac{t}{\log(1+t)}(Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t)))|\frac{1}{1+t}x^n\rangle \end{aligned} \tag{59}$$

It is easy to show that

$$\frac{1}{1+t}x^n = \sum_{l=0}^{\infty}(-t)^l x^n = \sum_{l=0}^{n}(-1)^l (n)_l x^{n-l}. \tag{60}$$

By (59) and (60), we get

$$\begin{aligned} C_n^{(k)} &= \frac{1}{n}\sum_{l=0}^{n}(-1)^l (n)_l \langle \frac{t}{\log(1+t)}(Lif_{k-1}(\log(1+t)) - Lif_k(\log(1+t)))|x^{n-l}\rangle \\ &= \frac{1}{n}\sum_{l=0}^{n}(-1)^{n-l}(n-l)!\binom{n}{l}(T_l^{(1,k-1)} - T_l^{(1,k)}). \end{aligned} \tag{61}$$

Therefore, by (61), we obtain the following lemma.

Lemma 2.6. *For $k \in \mathbb{Z}$, $n \geq 1$, we have*

$$C_n^{(k)} = \frac{1}{n}\sum_{l=0}^{n}(-1)^{n-l}(n-l)!\binom{n}{l}(T_l^{(1,k-1)} - T_l^{(1,k)}).$$

It is known that

$$\begin{aligned} &\partial_t^m Lif_k(\log(1+t)) \\ &= \sum_{a=1}^{m}\sum_{l=0}^{a} S_1(m,a)S_1(a+1,l+1)\frac{Lif_{k-l}(\log(1+t))}{(1+t)^m(\log(1+t))^a}, \ (see\ [11]). \end{aligned} \tag{62}$$

For $n \geq m \geq 1$, by (62), we get

$$\begin{aligned} C_n^{(k)} &= \langle Lif_k(\log(1+t))|x^n\rangle = \langle \partial_t^m Lif_k(\log(1+t))|x^{n-m}\rangle \\ &= \sum_{a=1}^{m}\sum_{l=0}^{a} S_1(m,a)S_1(a+1,l+1)\langle \frac{Lif_{k-l}(\log(1+t))}{(1+t)^m(\log(1+t))^a}|x^{n-m}\rangle \\ &= \sum_{a=1}^{m}\sum_{l=0}^{a} S_1(m,a)S_1(a+1,l+1)\frac{1}{(n-m+a)_a}\langle \frac{Lif_{k-l}(\log(1+t))}{(1+t)^m(\log(1+t))^a}|t^a x^{n-m+a}\rangle \\ &= \sum_{a=1}^{m}\sum_{l=0}^{a} S_1(m,a)S_1(a+1,l+1)\frac{1}{(n-m+a)_a} \\ &\qquad \times \langle \left(\frac{t}{\log(1+t)}\right)^a Lif_{k-l}(\log(1+t))|\frac{1}{(1+t)^m}x^{n-m+a}\rangle. \end{aligned} \tag{63}$$

Now, we observe that

$$\frac{1}{(1+t)^m}x^{n-m+a}=\sum_{s=0}^{n-m+a}(-1)^s\binom{m+s-1}{s}(n-m+a)_s x^{n-m+a-s}. \quad (64)$$

From (63) and (64), we have

$$\begin{aligned} C_n^{(k)}=\sum_{a=1}^{m}\sum_{l=0}^{a}\sum_{s=0}^{n-m+a}(-1)^s\binom{m+s-1}{s}\frac{(n-m)!}{(n-m+a-s)!} \\ \times S_1(m,a)S_1(a+1,l+1)T_{n-m+a-s}^{(a,k-l)}. \end{aligned} \quad (65)$$

Therefore, by (65), we obtain the following lemma.

Lemma 2.7. *For $n \geq m \geq 1$, we have*

$$\begin{aligned} C_n^{(k)}=\sum_{a=1}^{m}\sum_{l=0}^{a}\sum_{s=0}^{n-m+a}(-1)^s\binom{m+s-1}{s}\frac{(n-m)!}{(n-m+a-s)!} \\ \times S_1(m,a)S_1(a+1,l+1)T_{n-m+a-s}^{(a,k-l)}. \end{aligned}$$

For $S_n(x)\sim(g(t),f(t))$, it is known that

$$\frac{d}{dx}S_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\langle\bar{f}(t)|x^{n-l}\rangle S_l(x). \quad (66)$$

From (22) and (66), we have

$$\begin{aligned} \frac{d}{dx}C_n^{(k)}(x) &= \sum_{l=0}^{n-1}\binom{n}{l}\langle-\log(1+t)|x^{n-l}\rangle C_l^{(k)}(x) \\ &= \sum_{l=0}^{n-1}\binom{n}{l}\langle-\frac{\log(1+t)}{t}t|x^{n-l}\rangle C_l^{(k)}(x) \\ &= -\sum_{l=0}^{n-1}\binom{n}{l}(n-l)\langle\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m+1}|x^{n-l-1}\rangle C_l^{(k)}(x) \\ &= (-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)l!}C_l^{(k)}(x),\ (n\geq 1). \end{aligned} \quad (67)$$

For $C_n^{(k)}(x)\sim\left(\frac{1}{Lif_k(-t)},e^{-t}-1\right)$, $B_n^{(r)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^r,t\right)$, by (20) and (21), we have

$$C_n^{(k)}(x)=\sum_{m=0}^{n}C_{n,m}B_m^{(r)}(x), \quad (68)$$

where

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\langle \frac{\left(\frac{e^{-\log(1+t)}-1}{-\log(1+t)}\right)^r}{\frac{1}{Lif_k(\log(1+t))}}(-\log(1+t))^m | x^n\rangle \\ &= \frac{(-1)^m}{m!}\langle Lif_k(\log(1+t))\left(\frac{t}{(1+t)\log(1+t)}\right)^r(\log(1+t))^m | x^n\rangle \\ &= (-1)^m\sum_{l=0}^{n-m}\frac{1}{(l+m)!}S_1(l+m,m)(n)_{l+m} \\ &\quad\times\langle Lif_k(\log(1+t))\left(\frac{t}{(1+t)\log(1+t)}\right)^r | x^{n-l-m}\rangle. \end{aligned} \tag{69}$$

Carlitz's polynomials $\beta_n^{(r)}(x)$ are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x = \sum_{n=0}^{\infty}\beta_n^{(r)}(x)\frac{t^n}{n!}, \ (see\ [4,5]). \tag{70}$$

By (69) and (70), we get

$$C_{n,m} = (-1)^m\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\binom{n}{l+m}\binom{n-m-l}{a}S_1(l+m,m)\beta_a^{(r)}(-r)C_{n-m-l-a}^{(k)}. \tag{71}$$

Therefore, by (68) and (71), we obtain the following theorem.

Theorem 2.8. *For $k\in\mathbb{Z}$, $n\geq 0$, we have*

$$\begin{aligned} C_n^{(k)}(x) = \sum_{m=0}^{n}\Bigg\{(-1)^m\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\binom{n}{l+m}\binom{n-m-l}{a} \\ \times S_1(l+m,m)\beta_a^{(r)}(-r)C_{n-m-l-a}^{(k)}\Bigg\}B_n^{(r)}(x). \end{aligned}$$

Remark 1. It is known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{a=0}^{\infty}B_a^{(a)}\frac{t^a}{a!}. \tag{72}$$

Thus, by (72), we get

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^r = \sum_{a=0}^{\infty}\left(\sum_{a_1+\cdots+a_r=a}\binom{a}{a_1,\cdots,a_r}B_{a_1}^{(a_1)}\cdots B_{a_r}^{(a_r)}\right)\frac{t^a}{a!}. \tag{73}$$

From (69) and (73), we can also derive

$$C_{n,m} = (-1)^m \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{a_1+\cdots+a_r=a} \binom{n}{l+m} \binom{n-m-l}{a} \binom{a}{a_1, \cdots, a_r} \times S_1(l+m,m) B_{a_1}^{(a_1)} \cdots B_{a_r}^{(a_r)} C_{n-m-l-a}^{(k)}. \tag{74}$$

By (68) and (74), we get

$$C_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ (-1)^m \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{a_1+\cdots+a_r=a} \binom{n}{l+m} \binom{n-m-l}{a} \binom{a}{a_1, \cdots, a_r} \times S_1(l+m,m) B_{a_1}^{(a_1)} \cdots B_{a_r}^{(a_r)} C_{n-m-l-a}^{(k)} \right\} B_n^{(r)}(x). \tag{75}$$

From (3) and (22), we consider the following two Sheffer sequences:

$$C_n^{(k)}(x) \sim \left(\frac{1}{Lif_k(-t)}, e^{-t}-1 \right), \; H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda} \right)^r, t \right), \tag{76}$$

where $r \in \mathbb{Z}_{\geq 0}$.

Let us assume that

$$C_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(r)}(x|\lambda). \tag{77}$$

Then, by (21), we get

$$\begin{aligned} C_{n,m} &= \frac{(-1)^m}{m!(1-\lambda)^r} \langle Lif_k(\log(1+t)) \left(\frac{1}{1+t} - \lambda \right)^r |(\log(1+t))^m x^n \rangle \\ &= \frac{(-1)^m}{(1-\lambda)^r} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \sum_{a=0}^{r} \binom{r}{a} (-\lambda)^{r-a} \\ &\qquad \times \langle Lif_k(\log(1+t))(1+t)^{-a} | x^{n-m-l} \rangle \\ &= \frac{(-1)^m}{(1-\lambda)^r} \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \sum_{a=0}^{r} \binom{r}{a} (-\lambda)^{r-a} C_{n-m-l}^{(k)}(a). \end{aligned} \tag{78}$$

Therefore, by (77) and (78), we obtain the following theorem.

Theorem 2.9. *For $k \in \mathbb{Z}$, $n, r \in \mathbb{Z}_{\geq 0}$, we have*

$$C_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \frac{(-1)^{m+r}}{(1-\lambda)^r} \sum_{l=0}^{n-m} \sum_{a=0}^{r} (-1)^a \binom{n}{l+m} \binom{r}{a} \times S_1(l+m,m) \lambda^{r-a} C_{n-m-l}^{(k)}(a) \right\} H_m^{(r)}(x|\lambda).$$

Remark 2. By the same method, we can see that

$$C_n^{(k)}(x)=\sum_{m=0}^{n}\left\{\frac{(-1)^{m+r}}{(1-\lambda)^r}\sum_{l=0}^{n-m}\sum_{b=0}^{n-m-l}\sum_{a=0}^{r}(-1)^{a+b}\binom{n}{l+m}\binom{r}{a}\binom{a+b-1}{b}\right.$$
$$\left.\times(n-m-l)_b\lambda^{r-a}S_1(l+m,m)C_{n-m-l-b}^{(k)}\right\}H_m^{(r)}(x|\lambda). \tag{79}$$

For $C_n^{(k)}(x)\sim\left(\frac{1}{Lif_k(-t)},e^{-t}-1\right)$, $x^{(n)}\sim(1,1-e^{-t})$, we have

$$C_n^{(k)}(x)=\sum_{m=0}^{n}C_{n,m}x^{(m)}, \tag{80}$$

where

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\langle Lif_k(\log(1+t))(-t)^m|x^n\rangle\\&=\frac{(-1)^m}{m!}(n)_m\langle Lif_k(\log(1+t))|x^{n-m}\rangle\\&=(-1)^m\binom{n}{m}C_{n-m}^{(k)}.\end{aligned} \tag{81}$$

Therefore, by (77) and (78), we obtain the following theorem.

Theorem 2.10. *For $k\in\mathbb{Z}$, $n\geq 0$, we have*

$$C_n^{(k)}(x)=\sum_{m=0}^{n}(-1)^m\binom{n}{m}C_{n-m}^{(k)}x^{(m)},$$

where $x^{(n)}=x(x+1)\cdots(x+n-1)$.

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