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A Fixed Point Theorem in Complete Metric Spaces

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Abstract

In this paper we prove a fixed point theorem in three complete metric spaces. This result generalizes and unifies some of well-known fixed point theorems for metric spaces. Also the main theorem extends the Nesic's theorem from two metric spaces to three metric spaces.

Mathematics Subject Classification: 47H10, 54H25

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1. Introduction

Fisher [1], Popa [9] and Nešić′ [7] proved some fixed point theorems on two metric spaces. Nung [3], Jain et al [6] and Kikina [2] proved similar results for three metric spaces.

In this paper, using a new class of implicit relations, we prove a theorem as a corollary of which are taken the theorems: Fisher [2], Nung [3], Jain et al [6], Popa [9], Nešić′ [6], Kikina [2], etc.

In [3], [6] and [7] the following theorems are proved:

Theorem 1 (Nung [**Error! Reference source not found.**]) *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a continuous mapping of* X *into* Y, S *is a continuous mapping of* Y *into* Z *and* R *is a continuous mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx,RSy) \le c\max\{d(x,RSy),d(x,RSTx),\rho(y,Tx),\sigma(Sy,STx)\}$$

$$\rho(TRSy,TRz) \le c\max\{\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),d(Rz,RSy)\}$$

$$\sigma(STRz,STx) \le c\max\{\sigma(z,STx),\sigma(z,STRz),d(x,Rz),\rho(Tx,TRz)\}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. Then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

Theorem 2 (Jain et al. [7]) *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X *satisfying the inequalities*

$$\begin{aligned} d^2(RSy,RSTx) \le c\max\ \{&d(x,RSy)\rho(y,Tx),\rho(y,Tx)d(x,RSTx),\\ &d(x,RSTx)\sigma(Sy,STx),\sigma(Sy,STx)d(x,RSy)\}\end{aligned}$$

$$\begin{aligned} \rho^2(TRz,TRSy) \le c\max\ \{&\rho(y,TRz)\sigma(z,Sy),\sigma(z,Sy)\rho(y,TRSy),\\ &\rho(y,TRSy)d(Rz,RSy),d(Rz,RSy)\rho(y,TRz)\}\end{aligned}$$

$$\begin{aligned} \sigma^2(STx,STRz) \le c\max\ \{&\sigma(z,STx)d(x,Rz),d(x,Rz)\sigma(z,STRz),\\ &\sigma(z,STRz)\rho(Tx,TRz),\rho(Tx,TRz)\sigma(z,STx)\}\end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. If one of the mappings R,S,T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

Theorem 3 (Nešić [3]) *Let* (X,d) *and* (Y,ρ) *be complete metric spaces. Let* T *be a mapping of* X *into* Y *and* S *a mapping of* Y *into* X. *Denote*

$$M_1(x,y) = \{d^p(x,Sy),\rho^p(y,Tx),\rho^p(y,TSy)\}$$

and

$$M_2(x,y) = \{\rho^p(y,Tx),d^p(x,Sy),d^p(x,STx)\}$$

for all x in X, y in Y and $p = 1,2,3,\ldots$.

Let R^+ be the set of nonnegative real numbers, and let $F_i : R^+ \to R^+$ be a mapping such that $F_i(0) = 0$ and F_i is continuous at 0 for $i = 1,2$.

If T and S satisfying the inequalities

$$\rho^p(Tx,TSy) \le c_1 \max M_1(x,y) + F_1(\min M_1(x,y)),$$
$$d^p(Sy,STx) \le c_2 \max M_2(x,y) + F_2(\min M_2(x,y)),$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

2. Main results

We will prove a theorem which generalizes the Theorems Nung [3], Jain, Shrivastava and Fisher [6], Nešić′ [7] and extends the Theorem Nešić′ from two to three metric spaces. For this, we will use the implicit relations.

Let $\Phi_4^{(m)}$ be the set of continuous functions with 4 variables

$$\varphi : [0,+\infty)^4 \to [0,+\infty)$$

satisfying the properties:

φ is no decreasing in respect with each variable.

$\varphi(t,t,t,t) \le t^m, m \in N$.

Some examples of such functions are as follows:

Example 2.1 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$, *with* $m = 1$.

Example 2.2 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_i t_j : i, j \in I_4\}$, *with* $m = 2$.

Example 2.3 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1t_2, t_2t_3, t_3t_4, t_4t_1\}$, *with* $m = 2$.

Example 2.4 $\varphi(t_1,t_2,t_3,t_4) = \max\{t_1^p, t_2^p, t_3^p, t_4^p\}$, *with* $m = p$.

Let Ψ_4 be the set of continuous functions with 4 variables

$$\psi : [0,+\infty)^4 \to [0,+\infty)$$

satisfying the property

$$t_1t_2t_3t_4 = 0 \Rightarrow \psi(t_1,t_2,t_3,t_4) = 0.$$

Example 2.5 $\psi(t_1,t_2,t_3,t_4) = \min\{t_1,t_2,t_3,t_4\}$

Example 2.6 $\psi(t_1,t_2,t_3,t_4) = \min\{t_1^p, t_2^p, t_3^p, t_4^p\}, etc$

Let F be the set of continuous functions

$$F : [0,+\infty) \to [0,+\infty)$$

with $F(0) = 0$ (For example $F(t) = t^k, k > 0$).

Theorem 4 *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a*

mapping of Z *into* X *, such that at least one of them is a continuous mapping. Let* $\varphi_i \in \Phi_4^{(m)}, \psi_i \in \Psi_4, F_i \in \mathrm{F}$ *for* $i = 1,2,3$. *If there exists* $q \in [0,1)$ *and the following inequalities hold*

$$(1) \qquad d^m(RSy, RSTx) \le q\varphi_1(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)) + \\ + F_1(\psi_1(d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx))$$

$$(2) \qquad \rho^m(TRz, TRSy) \le q\varphi_2(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)) + \\ + F_2(\psi_2(\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy))$$

$$(3) \qquad \sigma^m(STx, STRz) \le q\varphi_3(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz)) + \\ + F_3(\psi_3(\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz))$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

Let $x_0 \in X$ be an arbitrary point. We define the sequences $(x_n), (y_n)$ and (z_n) in X, Y and Z respectively as follows:

$$x_n = (RST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n, n = 1,2,...$$

Denote

$$d_n = d(x_n, x_{n+1}), \rho_n = \rho(y_n, y_{n+1}), \sigma_n = \sigma(z_n, z_{n+1}), n = 1,2,...$$

By the inequality (2), for $y = y_n$ and $z = z_{n-1}$ we get:

$$\rho^m(y_n, y_{n+1}) \le q\varphi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)) + \\ + F_2(\psi_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)).$$

or

$$\rho_n^m \le q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) + F_2(\psi_2(0, \rho_n, \sigma_{n-1}, d_{n-1})) = \\ = q\varphi_2(0, \rho_n, \sigma_{n-1}, d_{n-1}) \qquad (4)$$

For the coordinates of the point $(0, \rho_n, \sigma_{n-1}, d_{n-1})$ we have:

$$\rho_n \le \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \qquad (5)$$

because, in case that $\rho_n > \max\{d_{n-1}, \sigma_{n-1}\}$ for some n, if we replace the coordinates with ρ_n and apply the property (b) of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\rho_n, \rho_n, \rho_n, \rho_n) \le q\rho_n^m.$$

This is impossible since $0 \le q < 1$.

By the inequalities (4), (5) and properties of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\max\{d_{n-1},\sigma_{n-1}\},\max\{d_{n-1},\sigma_{n-1}\},\max\{d_{n-1},\sigma_{n-1}\},\max\{d_{n-1},\sigma_{n-1}\}) \le$$
$$\le q\max\{d_{n-1}^m,\sigma_{n-1}^m\}.$$

Thus

$$\rho_n \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\} \tag{6}$$

By the inequality (3), for $x = x_{n-1}$ and $z = z_n$ we get:

$$\sigma^m(z_n,z_{n+1}) \le q\varphi_3(\sigma(z_n,z_n),\sigma(z_n,z_{n+1}),d(x_{n-1},x_n),\rho(y_n,y_{n+1})) +$$
$$+F_3(\psi_3(\sigma(z_n,z_n),\sigma(z_n,z_{n+1}),d(x_{n-1},x_n),\rho(y_n,y_{n+1})))$$

or

$$\sigma_n^m \le q\varphi_3(0,\sigma_n,d_{n-1},\rho_n) + F_3(0) = q\varphi_3(0,\sigma_n,d_{n-1},\rho_n) \tag{7}$$

In similiar way, we get:

$$\sigma_n^m \le q\max\{d_{n-1}^m,\rho_n^m\},\forall n\in N.$$

By this inequality and (6) we get:

$$\sigma_n \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\},\forall n\in N \tag{8}$$

By (1) for $x = x_n$ and $y = y_n$ we get:

$$d^m(x_n,x_{n+1}) \le q\varphi_1(d(x_n,x_n),d(x_n,x_{n+1}),\rho(y_n,y_{n+1}),\sigma(z_n,z_{n+1})) +$$
$$+F_1(\psi_1(d(x_n,x_n),d(x_n,x_{n+1}),\rho(y_n,y_{n+1}),\sigma(z_n,z_{n+1})))$$

or

$$d_n^m \le q\varphi_1(0,d_n,\rho_n,\sigma_n) + F(0) = q\varphi_1(0,d_n,\rho_n,\sigma_n) \tag{9}$$

For the same reasons we used to (5), for the coordinates of the point $(0,d_n,\rho_n,\sigma_n)$ we have:

$$d_n \le \max\{\rho_n,\sigma_n\},\forall n\in N.$$

Applying to (9) the properties of φ_1 and the inequalities (6), (8) we get:

$$d_n \le \sqrt[m]{q}\max\{\rho_n,\sigma_n\} \le \sqrt[m]{q}(\sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\}) =$$
$$= \sqrt[m]{q}(\sqrt[m]{q})\max\{d_{n-1},\sigma_{n-1}\} \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\}$$

or

$$d_n \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\} \tag{10}$$

By the inequalities (6), (8) and (10), using the mathematical induction, we get:

$$d(x_n, x_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\rho(y_n, y_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\sigma(z_n, z_{n+1}) \le r^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$

where $\sqrt[m]{q} = r < 1$.

Thus the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences. Since the metric spaces $(X, d), (Y, \rho)$ and (Z, σ) are complete metric spaces we have:

$$\lim_{n\to\infty} x_n = \alpha \in X, \lim_{n\to\infty} y_n = \beta \in Y, \lim_{n\to\infty} z_n = \gamma \in Z .$$

Assume that S is a continuous mapping. Then by

$$\lim_{x\to\infty} Sy_n = \lim_{x\to\infty} z_n .$$

it follows

$$S\beta = \gamma. \tag{11}$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta)) +$$
$$+F_1(\psi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta))).$$

By this inequality and (11) we get:

$$d^m(RS\beta, x_{n+1}) \le q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), 0) + F_1(0)$$

Letting n tend to infinity, we get

$$d^m(RS\beta, \alpha) \le q\varphi_1(d(RS\beta, \alpha), 0, 0, 0) \le qd^m(RS\beta, \alpha)$$

or

$$d(RS\beta, \alpha) = 0 \Leftrightarrow RS\beta = \alpha. \tag{12}$$

By (2), for $z = S\beta$ and $y = y_n$ we get:

$$\rho^m(TRS\beta, y_{n+1}) \le q\varphi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta)) +$$
$$+ F_2(\psi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta)))$$

Letting n tend to infinity and using (11), (12) we get:

$$\rho^m(TRS\beta, \beta) \le q\varphi_2(\rho(\beta, TRS\beta), 0, 0, 0) + F(0) .$$

or

$$\rho^m(TRS\beta, \beta) \le q\rho^m(\beta, TRS\beta) \Leftrightarrow TRS\beta = \beta \tag{13}$$

By (11), (12) and (13) it follows:

$$TRS\beta = TR\gamma = T\alpha = \beta$$
$$STR\gamma = ST\alpha = S\beta = \gamma$$
$$RST\alpha = RS\beta = \beta\gamma = \alpha$$

Thus, we proved that the points α, β, γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

Let we prove now the uniqueness of the fixed points α, β and γ.

Assume that there is α' a fixed point of RST different from α.

By (1) for $x = \alpha'$ and $y = T\alpha$ we get:

$$\begin{aligned} d^m(\alpha,\alpha') &= d^m(RST\alpha, RST\alpha') \le \\ &\le q\varphi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) + \\ &+F_1(\psi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha'))) = \\ &= q\varphi_1(d(\alpha',\alpha), 0, \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) + F(0) \le \\ &\le q\max\{d^m(\alpha',\alpha), \rho^m(T\alpha, T\alpha'), \sigma^m(ST\alpha, ST\alpha')\} \end{aligned}$$

or

$$d^m(\alpha,\alpha') = q\max A \qquad (14)$$

where $A = \{d^m(\alpha',\alpha); \rho^m(T\alpha, T\alpha'); \sigma^m(ST\alpha, ST\alpha')\}$.

We distinguish the following three cases:

Case I: If $\max A = d^m(\alpha',\alpha)$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le qd^m(\alpha',\alpha) \Leftrightarrow \alpha' = \alpha .$$

Case II: If $\max A = \rho^m(T\alpha, T\alpha')$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le q\rho^m(T\alpha, T\alpha') \qquad (15)$$

Continuing our argumentation for the Case 2, by (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$\begin{aligned} &\rho^m(T\alpha, T\alpha') = \rho^m(TRST\alpha, TRST\alpha') \le \\ &\le q\varphi_2(\rho(T\alpha', TRST\alpha), \rho(T\alpha', TRST\alpha'), \sigma(ST\alpha, ST\alpha'), d(RST\alpha', RST\alpha)) \\ &+F_2(\psi_2(\rho(T\alpha', TRST\alpha), \rho(T\alpha', TRST\alpha'), \sigma(ST\alpha, ST\alpha'), d(RST\alpha', RST\alpha))) \\ &= q\varphi_2(\rho(T\alpha', T\alpha), 0, \sigma(ST\alpha, ST\alpha'), d_1(\alpha,\alpha')) + F(0) \le q\max A \qquad (16) \end{aligned}$$

Since in Case II, $\max A = \rho^m(T\alpha, T\alpha')$, by (16) it follows

$$\rho^m(T\alpha, T\alpha') \le q\rho^m(T\alpha, T\alpha')$$

or

$$\rho(T\alpha, T\alpha') = 0.$$

By (15), it follows $d(\alpha, \alpha') = 0$.

Case III: If $\max A = \sigma^m(ST\alpha, ST\alpha')$, then by (14) it follows

$$d^m(\alpha, \alpha') \le q\sigma^m(ST\alpha, ST\alpha') \qquad (17)$$

By the inequality (3), for $x = RST\alpha, z = ST\alpha'$, in similar way we obtain:

$$\sigma^m(ST\alpha, ST\alpha') \le q \max A = q\sigma^m(ST\alpha, ST\alpha')$$

It follows

$$\sigma(ST\alpha, ST\alpha') = 0$$

and by (17) it follows

$$d(\alpha, \alpha') = 0$$

Thus, we have again $\alpha = \alpha'$.

In the same way, it is proved the uniqueness of β and γ.

3. Corollaries

Corollary 3.1 *Let $(X, d), (Y, \rho)$ and (Z, σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. Let $F:[0,+\infty) \to [0,+\infty)$ be continuous with $F(0) = 0$. If there exists $q \in [0,1)$ and $m \in N$ such that the following inequalities hold*

$$\begin{aligned}
d^m(RSy, RSTx) &\le q\max\{d^m(x, RSy), d^m(x, RSTx), \rho^m(y, Tx), \sigma^m(Sy, STx)\} + \\
&\quad + F(\min\{d^m(x, RSy), d^m(x, RSTx), \rho^m(y, Tx), \sigma^m(Sy, STx)\} \\
\rho^m(TRz, TRSy) &\le q\max\{\rho^m(y, TRz), \rho^m(y, TRSy), \sigma^m(z, Sy), d^m(Rz, RSy)) + \\
&\quad + F(\min\{\rho^m(y, TRz), \rho^m(y, TRSy), \sigma^m(z, Sy), d^m(Rz, RSy)\} \\
\sigma^m(STx, STRz) &\le q\max\{\sigma^m(z, STx), \sigma^m(z, STRz), d^m(x, Rz), \rho^m(Tx, TRz)\} + \\
&\quad + F(\min\{\sigma^m(z, STx), \sigma^m(z, STRz), d^m(x, Rz), \rho^m(Tx, TRz)\}
\end{aligned}$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$.

Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

The proof follows by Theorem 2.6 in the case $F_1 = F_2 = F_3 = F, \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(m)}$ such that $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1^m, t_2^m, t_3^m, t_4^m\}$ and $\psi_1 = \psi_2 = \psi_3 = \psi$, where $\psi(t_1, t_2, t_3, t_4) = \min\{t_1^m, t_2^m, t_3^m, t_4^m\}$.

Corollary 3.1 extends Theorem 1.3 (Nešić [7]) from two in three metric spaces.

Corollary 3.2 *Theorem 1.1 (Nung [3]) is taken by Corollary 3.1 for* $m = 1$ *and* $F = 0$.

Corollary 3.3 *Theorem 1.2 (Jain et. al. [6]) is taken by Theorem 2.6 in case* $F_1 = F_2 = F_3 = 0; \varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_4^{(2)}$ *such that* $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1t_3, t_2t_3, t_2t_4, t_4t_1\}$.

Corollary 3.4 *Theorem Kikina (Theorem 2.1, [2]) is taken by Corollary 3.1 in case* $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1^m, t_2^m, t_3^m\}$ *and* $\psi(t_1, t_2, t_3, t_4) = \min\{t_1^m, t_2^m, t_3^m\}$.

Corollary 3.5 *Let* $(X, d), (Y, \rho)$ *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z. $\varphi_i \in \Phi_3, \psi_1 \in \Psi, F_i \in \mathrm{F}$ *for* $i = 1, 2$. *If there exists* $q \in [0,1)$ *such that the following inequalities hold*

$$(1') \quad d^m(Sy, STx) \le q\varphi_1(d(x, Sy), d(x, STx), \rho(y, Tx)) + \\ + F_1(\psi_1(d(x, Sy), d(x, STx), \rho(y, Tx))$$

$$(2') \quad \rho^m(Tx, TSy) \le q\varphi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy)) + \\ + F_2(\psi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy))$$

for all $x \in X, y \in Y$, then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further, $T\alpha = \beta, S\beta = \gamma$.

By Theorem 2.6, if we take: $Z = X, \sigma = d$ the mapping R as the identity mapping in X, $\varphi_i(t_1, t_2, t_3, t_4) = \varphi_i(t_1, t_2, t_3), \psi_i(t_1, t_2, t_3, t_4) = \psi_i(t_1, t_2, t_3)$, then the inequality (1) takes the form (1'), the inequality (2) takes the form (2') and the inequality (3) is always satisfied since his left side is $\sigma^m(STx, STx) = 0$. Thus, the satisfying of the conditions (1), (2) and (3) is reduced in satisfying of the conditions (1') and (2').

The mappings T and S may be not continuous, while from the mappings T, S and R for which we applied Theorem 2.6, the identity mapping R is continuous. This completes the proof.

We have the following corollary.

Corollary 3.6 *(Theorem Nešić [7]). Theorem 1.3 is taken by Corollary 3.5 for* $\varphi_1 = \varphi_2 = \varphi; \psi_1 = \psi_2 = \psi$ *such that* $\varphi(t_1, t_2, t_3) = \max\{t_1^m, t_2^m, t_3^m\}$ *and*

$\psi(t_1, t_2, t_3) = \min\{t_1^m, t_2^m, t_3^m\}$.

We emphasize the fact that in the Theorem 1.3, the mappings F_1 and F_2 can be replaced by $F(t) = \max\{F_1(t), F_2(t)\}$ and c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Corollary 5 *Theorem Popa (Theorem 2, [9]) is taken by Corollary 3.5 for* $\varphi_1 = \varphi_2 = \varphi$ *such that* $\varphi(t_1, t_2, t_3) = \max\{t_1t_2, t_1t_3, t_2t_3\}$ *with* $m = 2$ *and* $F = 0$.

We also emphasize here that the constants c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Remark. As corollaries of these results we can obtain other propositions determined by the form of implicit functions, for example Proposition Popa (Corollary 2, [9]), Theorem Fisher (Theorem 1, [1]) etc.

References

[1] B. Fisher, *Fixed point in two metric spaces*, Glasnik Matem. 16(36) (1981), 333-337.

[2] L. Kikina, *Fixed points theorems in three metric spaces*, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 13-16, 619-626.

[3] N. P. Nung, *A fixed point theorem in three metric spaces*, Math. Sem. Notes, Kobe. Univ. 11 (1983), 77-79.

[4] R. K. Jain, H. K. Sahu, B. Fisher, *A related fixed point theorem on three metric spaces*, Kyungpook Math. J. 36 (1996), 151-154.

[5] R. K. Jain, H. K. Sahu, B. Fisher, *Related fixed point theorems to three metric spaces*, Novi Sad J. Math., Vol. 26, No. 1, (1996), 11-17.

[6] R. K. Jain, A. K. Shrivastava, B. Fisher, *Fixed points on three complete metric spaces*, Novi Sad J. Math. Vol. 27, No. 1 (1997), 27-35.

[7] S. Č. Nešić′, *A fixed point theorem in two metric spaces*, Bull. Math. Soc. Sci. Math. Roumanie, Tome 44(94) (2001), No.3, 253-257.

[8] S. Č. Nešić′, *Common fixed point theorems in metric spaces*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 46(94) (200 3), No.3-4, 149-155.

[9] V. Popa, *Fixed points on two complete metric spaces*, Zb. Rad. Prirod.-Mat. Fak. (N.S.) Ser. Mat. **21(1)** (1991), 83-93.

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