A Fixed Point Theorem in Complete Metric Spaces

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Abstract

In this paper we prove a fixed point theorem in three complete metric spaces. This result generalizes and unifies some of well-known fixed point theorems for metric spaces. Also the main theorem extends the Nesic’s theorem from two metric spaces to three metric spaces.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Cauchy sequence, fixed point, complete metric space, implicit relations

1. Introduction


In this paper, using a new class of implicit relations, we prove a theorem as a corollary of which are taken the theorems: Fisher [2], Nung [3], Jain et al [6], Popa [9], Nesic’ [6], Kikina [2], etc.
In [3], [6] and [7] the following theorems are proved:

**Theorem 1** (Nung [Error! Reference source not found.]) Let \((X, d), (Y, \rho)\) and \((Z, \sigma)\) be complete metric spaces and suppose \(T\) is a continuous mapping of \(X\) into \(Y\), \(S\) is a continuous mapping of \(Y\) into \(Z\) and \(R\) is a continuous mapping of \(Z\) into \(X\) satisfying the inequalities

\[
d(RSTx, RSy) \leq c \max \{d(x, RSy), d(x, RSTx), \rho(y, Tx), \sigma(Sy, STx)\}
\]
\[
\rho(TRSy, TRz) \leq c \max \{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy)\}
\]
\[
\sigma(STRz, STx) \leq c \max \{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\}
\]

for all \(x\) in \(X\), \(y\) in \(Y\) and \(z\) in \(Z\), where \(0 < c < 1\). Then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\) and \(STR\) has a unique fixed point \(w\) in \(Z\). Further, \(Tu = v, Sv = w\) and \(Rw = u\).

**Theorem 2** (Jain et al. [7]) Let \((X, d), (Y, \rho)\) and \((Z, \sigma)\) be complete metric spaces and suppose \(T\) is a continuous mapping of \(X\) into \(Y\), \(S\) is a continuous mapping of \(Y\) into \(Z\) and \(R\) is a continuous mapping of \(Z\) into \(X\) satisfying the inequalities

\[
d^2(RSy, RSTx) \leq c \max \{d(x, RSy, y), \rho(y, Tx), d(x, RSTx)\}
\]
\[
d(x, RSTx) \sigma(Sy, STx), \sigma(Sy, STx) d(x, RSy)\}
\]
\[
\rho^2(TRz, TRSy) \leq c \max \{\rho(y, TRz), \sigma(z, Sy), \sigma(y, TRSy), \rho(y, TRSy), \rho(y, TRz)\}
\]
\[
\rho^2(Sy, STRz) \leq c \max \{\sigma(z, STx), d(x, Rz), d(x, Rz) \sigma(z, STRz), \sigma(z, STRz) \rho(Tx, TRz), \rho(Tx, TRz) \sigma(z, STx)\}
\]

for all \(x\) in \(X\), \(y\) in \(Y\) and \(z\) in \(Z\), where \(0 \leq c < 1\). If one of the mappings \(R, S, T\) is continuous, then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\) and \(STR\) has a unique fixed point \(w\) in \(Z\). Further, \(Tu = v, Sv = w\) and \(Rw = u\).

**Theorem 3** (Nešić [3]) Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces. Let \(T\) be a mapping of \(X\) into \(Y\) and \(S\) a mapping of \(Y\) into \(X\). Denote

\[
M_1(x, y) = \{d^p(x, Sy), \rho^p(y, Tx), \rho^p(y, TSy)\}
\]

and

\[
M_2(x, y) = \{\rho^p(y, Tx), d^p(x, Sy), d^p(x, STx)\}
\]

for all \(x\) in \(X\), \(y\) in \(Y\) and \(p = 1, 2, 3, \ldots\).

Let \(R^+\) be the set of nonnegative real numbers, and let \(F_i : R^+ \to R^+\) be a mapping such that \(F_i(0) = 0\) and \(F_i\) is continuous at 0 for \(i = 1, 2\),

If \(T\) and \(S\) satisfying the inequalities
$\rho^\phi(Tx, TSy) \leq c_1 \max M_1(x, y) + F^\phi(\min M_1(x, y)),$

$d^\rho(Sy, STx) \leq c_2 \max M_2(x, y) + F^\sigma(\min M_2(x, y)),$

for all $x$ in $X$ and $y$ in $Y$, where $0 \leq c_1, c_2 < 1$, then $ST$ has a unique fixed point $z$ in $X$ and $TS$ has a unique fixed point $w$ in $Y$. Further, $Tz = w$ and $Sw = z$.

\section{Main results}

We will prove a theorem which generalizes the Theorems Nung [3], Jain, Shrivastava and Fisher [6], Nešić [7] and extends the Theorem Nešić from two to three metric spaces. For this, we will use the implicit relations.

Let $\Phi_4$ be the set of continuous functions with 4 variables

$\varphi : [0, \infty)^4 \rightarrow [0, \infty)$

satisfying the properties:

$\varphi$ is no decreasing in respect with each variable.

$\varphi(t, t, t, t) \leq t^m, m \in \mathbb{N}$.

Some examples of such functions are as follows:

**Example 2.1** $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, with $m = 1$.

**Example 2.2** $\varphi(t_1, t_2, t_3, t_4) = \max\{t_i, t_j : i, j \in I_4\}$, with $m = 2$.

**Example 2.3** $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$, with $m = 2$.

**Example 2.4** $\varphi(t_1, t_2, t_3, t_4) = \max\{t_1^p, t_2^p, t_3^p, t_4^p\}$, with $m = p$.

Let $\Psi_4$ be the set of continuous functions with 4 variables

$\psi : [0, \infty)^4 \rightarrow [0, \infty)$

satisfying the property

$t_1 t_2 t_3 t_4 = 0 \Rightarrow \psi(t_1, t_2, t_3, t_4) = 0$.

**Example 2.5** $\psi(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\}$

**Example 2.6** $\psi(t_1, t_2, t_3, t_4) = \min\{t_1^p, t_2^p, t_3^p, t_4^p\}$, etc.

Let $F$ be the set of continuous functions

$F : [0, \infty) \rightarrow [0, \infty)$

with $F(0) = 0$ (For example $F(t) = t^k, k > 0$).

**Theorem 4** Let $(X, d), (Y, \rho)$ and $(Z, \sigma)$ be complete metric spaces and suppose $T$ is a mapping of $X$ into $Y$, $S$ is a mapping of $Y$ into $Z$ and $R$ is a
mapping of $Z$ into $X$, such that at least one of them is a continuous mapping. Let
$\phi_i \in \Phi_{4(i)}, \psi_i \in \Psi_i, F_i \in F$ for $i = 1, 2, 3$. If there exists $q \in [0,1)$ and the following
inequalities hold

\begin{align}
(1) \quad d^n(RS_y, RST_x) & \leq q\phi_i(d(x, RS_y), d(x, RST_x), \rho(y, Tx), \sigma(Sy, STx)) + \\
& + F_1(\psi_i(d(x, RS_y), d(x, RST_x), \rho(y, Tx), \sigma(Sy, STx))
\end{align}

\begin{align}
(2) \quad \rho^n(TRz, TRSy) & \leq q\phi_i(\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy)) + \\
& + F_2(\psi_i(\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy))
\end{align}

\begin{align}
(3) \quad \sigma^n(STx, STRz) & \leq q\phi_i(\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)) + \\
& + F_3(\psi_i(\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz))
\end{align}

for all $x \in X, y \in Y$ and $z \in Z$, then $RST$ has a unique fixed point $\alpha \in X$, $TRS$ has a unique fixed point
$\beta \in Y$ and $STR$ has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

Let $x_0 \in X$ be an arbitrary point. We define the sequences $(x_n), (y_n)$ and $(z_n)$ in $X, Y$ and $Z$ respectively as follows:

$$x_n = (RST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n, n = 1, 2,...$$

Denote

$$d_n = d(x_n, x_{n+1}), \rho_n = \rho(y_n, y_{n+1}), \sigma_n = \sigma(z_n, z_{n+1}), n = 1, 2,...$$

By the inequality (2), for $y = y_n$ and $z = z_{n-1}$ we get:

\begin{align}
\rho^n(y_n, y_{n+1}) & \leq q\phi_i(\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)) + \\
& + F_2(\psi_i(\rho(y_n, y_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), d(x_{n-1}, x_n)).
\end{align}

or

$$\rho_n^m \leq q\phi_i(0, \rho_n, \sigma_{n-1}, d_{n-1}) + F_2(\psi_i(0, \rho_n, \sigma_{n-1}, d_{n-1})) = \\
= q\phi_i(0, \rho_n, \sigma_{n-1}, d_{n-1})$$

(4)

For the coordinates of the point $(0, \rho_n, \sigma_{n-1}, d_{n-1})$ we have:

$$\rho_n \leq \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N$$

(5)

because, in case that $\rho_n > \max\{d_{n-1}, \sigma_{n-1}\}$ for some $n$, if we replace the coordinates with $\rho_n$ and apply the property (b) of $\phi_2$ we get:

$$\rho_n^m \leq q\phi_2(\rho_n, \rho_n, \rho_n, \rho_n) \leq q\rho_n^m.$$  

This is impossible since $0 \leq q < 1$.

By the inequalities (4), (5) and properties of $\phi_2$ we get:
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\[ \rho_n^m \leq q \varphi_n^m \left( \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}, \max\{d_{n-1}, \sigma_{n-1}\}\right) \leq q \max\{d_{n-1}, \sigma_{n-1}\}. \]

Thus

\[ \rho_n \leq q \max\{d_{n-1}, \sigma_{n-1}\} \] (6)

By the inequality (3), for \( x = x_{n-1} \) and \( z = z_n \) we get:

\[ \sigma^m(z_n, z_{n+1}) \leq q \varphi_n^m(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1})) + F_3(\psi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n), \rho(y_n, y_{n+1}))) \]

or

\[ \sigma_n^m \leq q \varphi_n^m(0, \sigma_n, d_{n-1}, \rho_n) + F_3(0) = q \varphi_n^m(0, \sigma_n, d_{n-1}, \rho_n) \] (7)

In similar way, we get:

\[ \sigma_n^m \leq q \max\{d_{n-1}, \rho_n^m\}, \forall n \in N. \]

By this inequality and (6) we get:

\[ \sigma_n \leq \sqrt[q]{q} \max\{d_{n-1}, \sigma_{n-1}\}, \forall n \in N \] (8)

By (1) for \( x = x_n \) and \( y = y_n \) we get:

\[ d^m(x_n, x_{n+1}) \leq q \varphi_n^m(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1})) + F_1(\psi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}))) \]

or

\[ d_n^m \leq q \varphi_n^m(0, d_n, \rho_n, \sigma_n) + F(0) = q \varphi_n^m(0, d_n, \rho_n, \sigma_n) \] (9)

For the same reasons we used to (5), for the coordinates of the point \( (0, d_n, \rho_n, \sigma_n) \) we have:

\[ d_n \leq \max\{\rho_n, \sigma_n\}, \forall n \in N. \]

Applying to (9) the properties of \( \varphi_n \) and the inequalities (6), (8) we get:

\[ d_n \leq \sqrt[q]{q} \max\{\rho_n, \sigma_n\} \leq \sqrt[q]{q} \left( \sqrt[q]{q} \max\{d_{n-1}, \sigma_{n-1}\} \right) = \]

\[ = \sqrt[q]{q} \left( \sqrt[q]{q} \right) \max\{d_{n-1}, \sigma_{n-1}\} \leq \sqrt[q]{q} \max\{d_{n-1}, \sigma_{n-1}\} \]

or

\[ d_n \leq \sqrt[q]{q} \max\{d_{n-1}, \sigma_{n-1}\} \] (10)

By the inequalities (6), (8) and (10), using the mathematical induction, we get:
\[
d(x_n, x_{n+1}) \leq r^{n-1} \max \{d(x_1, x_2), \sigma(z_1, z_2)\}
\]
\[
\rho(y_n, y_{n+1}) \leq r^{n-1} \max \{d(x_1, x_2), \sigma(z_1, z_2)\}
\]
\[
\sigma(z_n, z_{n+1}) \leq r^{n-1} \max \{d(x_1, x_2), \sigma(z_1, z_2)\}
\]

where \( \sqrt[n]{q} = r < 1 \).

Thus the sequences \((x_n), (y_n)\) and \((z_n)\) are Cauchy sequences. Since the metric spaces \((X, d), (Y, \rho)\) and \((Z, \sigma)\) are complete metric spaces we have:

\[
\lim_{n \to \infty} x_n = \alpha \in X, \lim_{n \to \infty} y_n = \beta \in Y, \lim_{n \to \infty} z_n = \gamma \in Z.
\]

Assume that \(S\) is a continuous mapping. Then by

\[
\lim_{x \to \infty} Sx_n = \lim_{x \to \infty} z_n.
\]

it follows

\[
S\beta = \gamma. \quad (11)
\]

By (1), for \(y = \beta\) and \(x = x_n\) we get:

\[
d^n(RS\beta, x_{n+1}) \leq q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta)) + F_1(\psi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), \sigma(\gamma, S\beta))).
\]

By this inequality and (11) we get:

\[
d^n(RS\beta, x_{n+1}) \leq q\varphi_1(d(x_n, RS\beta), d(x_n, x_{n+1}), \rho(\beta, y_{n+1}), 0) + F_1(0)
\]

Letting \(n\) tend to infinity, we get

\[
d^n(RS\beta, \alpha) \leq q\varphi_1(d(RS\beta, \alpha), 0, 0, 0) \leq qd^n(\alpha)\rho(\beta, y_{n+1})\alpha
\]

or

\[
d(RS\beta, \alpha) = 0 \Leftrightarrow RS\beta = \alpha. \quad (12)
\]

By (2), for \(z = S\beta\) and \(y = y_n\) we get:

\[
\rho^n(TRS\beta, y_{n+1}) \leq q\varphi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta)) + F_2(\psi_2(\rho(y_n, TRS\beta), \rho(y_n, y_{n+1}), \sigma(S\beta, z_n), d(x_n, RS\beta)))
\]

Letting \(n\) tend to infinity and using (11), (12) we get:

\[
\rho^n(TRS\beta, \beta) \leq q\varphi_2(\rho(\beta, TRS\beta), 0, 0, 0) + F(0).
\]

or

\[
\rho^n(TRS\beta, \beta) \leq q\rho^n(\beta, TRS\beta) \Leftrightarrow TRS\beta = \beta. \quad (13)
\]
By (11), (12) and (13) it follows:

\[ TRS\beta = TR\gamma = T\alpha = \beta \]
\[ STR\gamma = ST\alpha = S\beta = \gamma \]
\[ RST\alpha = RS\beta = B\gamma = \alpha \]

Thus, we proved that the points \( \alpha, \beta, \gamma \) are fixed points of \( RST, TRS \) and \( STR \) respectively.

In the same conclusion we would arrive if one of the mappings \( R \) or \( T \) would be continuous.

Let we prove now the uniqueness of the fixed points \( \alpha, \beta \) and \( \gamma \).

Assume that there is \( \alpha' \) a fixed point of \( RST \) different from \( \alpha \).

By (1) for \( x = \alpha' \) and \( y = T\alpha \) we get:

\[
d^m(\alpha, \alpha') = d^m(RST\alpha, RST\alpha') \leq \\
\leq q\varphi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) + \\
+ F_1(\psi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) = \\
= q\varphi_1(d(\alpha', \alpha), 0, \rho(T\alpha, T\alpha'), \sigma(ST\alpha, ST\alpha')) + F(0) \leq \\
\leq q \max \{d^m(\alpha', \alpha), \rho^m(T\alpha, T\alpha'), \sigma^m(ST\alpha, ST\alpha')\}
\]

or

\[
d^m(\alpha, \alpha') = q \max A \tag{14}
\]

where \( A = \{d^m(\alpha', \alpha); \rho^m(T\alpha, T\alpha'); \sigma^m(ST\alpha, ST\alpha')\} \).

We distinguish the following three cases:

Case I: If \( \max A = d^m(\alpha', \alpha) \), then the inequality (14) implies

\[
d^m(\alpha, \alpha') \leq q d^m(\alpha', \alpha) \iff \alpha' = \alpha.
\]

Case II: If \( \max A = \rho^m(T\alpha, T\alpha') \), then the inequality (14) implies

\[
d^m(\alpha, \alpha') \leq q\rho^m(T\alpha, T\alpha') \tag{15}
\]

Continuing our argumentation for the Case 2, by (2) for \( z = ST\alpha \) and \( y = T\alpha' \) we have:

\[
\rho^m(T\alpha, T\alpha') = \rho^m(TRST\alpha, TRST\alpha') \leq \\
\leq q\varphi_2(\rho(T\alpha', TRST\alpha), \rho(T\alpha', TRST\alpha'), \sigma(ST\alpha, ST\alpha'), d(RST\alpha', RST\alpha)) + \\
+ F_2(\psi_2(\rho(T\alpha', TRST\alpha), \rho(T\alpha', TRST\alpha'), \sigma(ST\alpha, ST\alpha'), d(RST\alpha', RST\alpha)) = \\
= q\varphi_2(\rho(T\alpha', T\alpha), 0, \sigma(ST\alpha, ST\alpha'), d_1(\alpha, \alpha')) + F(0) \leq q \max A \tag{16}
\]
Since in Case II, \( \max A = \rho^n(T\alpha, T\alpha') \), by (16) it follows
\[
\rho^n(T\alpha, T\alpha') \leq q\rho^n(T\alpha, T\alpha')
\]
or
\[
\rho(T\alpha, T\alpha') = 0.
\]
By (15), it follows \( d(\alpha, \alpha') = 0 \).

**Case III:** If \( \max A = \sigma^n(ST\alpha, ST\alpha') \), then by (14) it follows
\[
d^n(\alpha, \alpha') \leq q\sigma^n(ST\alpha, ST\alpha')
\]
By the inequality (3), for \( x = RST\alpha, z = ST\alpha' \), in similar way we obtain:
\[
\sigma^n(ST\alpha, ST\alpha') \leq q \max A = q\sigma^n(ST\alpha, ST\alpha')
\]
It follows
\[
\sigma(ST\alpha, ST\alpha') = 0
\]
and by (17) it follows
\[
d(\alpha, \alpha') = 0
\]
Thus, we have again \( \alpha = \alpha' \).
In the same way, it is proved the uniqueness of \( \beta \) and \( \gamma \).

### 3. Corollaries

**Corollary 3.1** Let \((X,d),(Y,\rho)\) and \((Z,\sigma)\) be complete metric spaces and suppose \( T \) is a mapping of \( X \) into \( Y \), \( S \) is a mapping of \( Y \) into \( Z \) and \( R \) is a mapping of \( Z \) into \( X \), such that at least one of them is a continuous mapping. Let \( F : [0, +\infty) \to [0, +\infty) \) be continuous with \( F(0) = 0 \). If there exists \( q \in [0,1) \) and \( m \in \mathbb{N} \) such that the following inequalities hold
\[
d^n(RSy, RSTx) \leq q \max \{d^n(x, RSy), d^n(x, RSTx), \rho^n(y, Tx), \sigma^n(Sy, STx)\} +
F(\min \{d^n(x, RSy), d^n(x, RSTx), \rho^n(y, Tx), \sigma^n(Sy, STx)\})
\]
\[
\rho^n(TRz, TRSy) \leq q \max \{\rho^n(y, TRz), \rho^n(y, TRSy), \sigma^n(z, Sy), d^n(Rz, RSy)\} +
F(\min \{\rho^n(y, TRz), \rho^n(y, TRSy), \sigma^n(z, Sy), d^n(Rz, RSy)\})
\]
\[
\sigma^n(STx, STRz) \leq q \max \{\sigma^n(z, STx), \sigma^n(z, STRz), d^n(x, Rz), \rho^n(Tx, TRz)\} +
F(\min \{\sigma^n(z, STx), \sigma^n(z, STRz), d^n(x, Rz), \rho^n(Tx, TRz)\})
\]
for all \( x \in X, y \in Y \) and \( z \in Z \), then \( RST \) has a unique fixed point \( \alpha \in X \), \( TRS \) has a unique fixed point \( \beta \in Y \) and \( STR \) has a unique fixed point \( \gamma \in Z \).
Further, \( T\alpha = \beta, S\beta = \gamma \) and \( R\gamma = \alpha \).

The proof follows by Theorem 2.6 in the case \( F_1 = F_2 = F_3 = F, \phi_1 = \phi_2 = \phi_3 = \phi \in \Phi_4^{(m)} \) such that \( \phi(t_1, t_2, t_3, t_4) = \max\{t_1^m, t_2^m, t_3^m, t_4^m\} \) and \( \psi_1 = \psi_2 = \psi_3 = \psi \), where \( \psi(t_1, t_2, t_3, t_4) = \min\{t_1^m, t_2^m, t_3^m, t_4^m\} \).

Corollary 3.1 extends Theorem 1.3 (Nešić [7]) from two in three metric spaces.

**Corollary 3.2** Theorem 1.1 (Nung [3]) is taken by Corollary 3.1 for \( m = 1 \) and \( F = 0 \).

**Corollary 3.3** Theorem 1.2 (Jain et. al. [6]) is taken by Theorem 2.6 in case \( F_1 = F_2 = F_3 = F \); \( \phi_1 = \phi_2 = \phi_3 = \phi \in \Phi_4^{(2)} \) such that
\[
\phi(t_1, t_2, t_3, t_4) = \max\{t_1^m, t_2^m, t_3^m\}.
\]

**Corollary 3.4** Theorem Kikina (Theorem 2.1, [2]) is taken by Corollary 3.1 in case
\[
\phi(t_1, t_2, t_3, t_4) = \max\{t_1^m, t_2^m, t_3^m\} \quad \text{and} \quad \psi(t_1, t_2, t_3, t_4) = \min\{t_1^m, t_2^m, t_3^m\}.
\]

**Corollary 3.5** Let \((X, d), (Y, \rho)\) be complete metric spaces and suppose \( T \) is a mapping of \( X \) into \( Y \), \( S \) is a mapping of \( Y \) into \( Z \). \( \phi_i \in \Phi_i, \psi_i \in \Psi_i, F_i \in F \) for \( i = 1, 2 \). If there exists \( q \in [0, 1) \) such that the following inequalities hold
\[
(1') \quad d^n(Sy, STx) \leq q\phi_1(d(x, Sy), d(x, STx), \rho(y, Tx)) + F_1(\psi_1(d(x, Sy), d(x, STx), \rho(y, Tx)))
\]
\[
(2') \quad \rho^n(Tx, TSy) \leq q\phi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy)) + F_2(\psi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy)))
\]
for all \( x \in X, y \in Y \), then \( ST \) has a unique fixed point \( \alpha \in X \) and \( TS \) has a unique fixed point \( \beta \in Y \). Further, \( T\alpha = \beta, S\beta = \gamma \).

By Theorem 2.6, if we take: \( Z = X, \sigma = d \) the mapping \( R \) as the identity mapping in \( X \), \( \phi_i(t_1, t_2, t_3, t_4) = \phi_i(t_1, t_2, t_3), \psi_i(t_1, t_2, t_3, t_4) = \psi_i(t_1, t_2, t_3) \), then the inequality (1) takes the form \( (1') \), the inequality (2) takes the form \( (2') \) and the inequality (3) is always satisfied since his left side is \( \sigma^n(STx, STx) = 0 \). Thus, the satisfying of the conditions (1), (2) and (3) is reduced in satisfying of the conditions \( (1') \) and \( (2') \).

The mappings \( T \) and \( S \) may be not continuous, while from the mappings \( T, S \) and \( R \) for which we applied Theorem 2.6, the identity mapping \( R \) is continuous. This completes the proof.

We have the following corollary.

**Corollary 3.6** (Theorem Nešić [7]). Theorem 1.3 is taken by Corollary 3.5 for \( \phi_1 = \phi_2 = \phi, \psi_1 = \psi_2 = \psi \) such that \( \phi(t_1, t_2, t_3) = \max\{t_1^m, t_2^m, t_3^m\} \) and
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\[ \psi(t_1, t_2, t_3) = \min\{t_1^m, t_2^m, t_3^m\}. \]

We emphasize the fact that in the Theorem 1.3, the mappings \( F_1 \) and \( F_2 \) can be replaced by \( F(t) = \max\{F_1(t), F_2(t)\} \) and \( c_1, c_2 \) can be replaced by \( q = \max\{c_1, c_2\} \).

**Corollary 5** Theorem Popa (Theorem 2, [9]) is taken by Corollary 3.5 for \( \varphi_1 = \varphi_2 = \varphi \) such that \( \varphi(t_1, t_2, t_3) = \max\{t_1, t_1, t_1\} \) with \( m = 2 \) and \( F = 0 \).

We also emphasize here that the constants \( c_1, c_2 \) can be replaced by \( q = \max\{c_1, c_2\} \).

**Remark.** As corollaries of these results we can obtain other propositions determined by the form of implicit functions, for example Proposition Popa (Corollary 2, [9]), Theorem Fisher (Theorem 1, [1]) etc.

**References**


Received: June 16, 2013