Soliton Solutions for Wick-type Stochastic Fractional KdV Equations

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Abstract

Wick-type stochastic fractional KdV equations are researched. A fractional Bäcklund transformation (FBT) is derived via the homogeneous balance principle. By using this FBT, Hermit transform and white noise theory, we obtain stochastic soliton solutions for the Wick-type stochastic fractional KdV equations.

Keywords: Stochastic KdV equations; Fractional calculus; Soliton; White noise

1 Introduction

In this paper, we consider the Wick-type stochastic fractional KdV equations as the following form:

\[ D_t^\alpha U + a(t)U \cdot D_x^\alpha U + b(t)D_x^{3\alpha}U = W(t) \cdot R^\circ(U, D_x^\alpha U, D_x^{2\alpha}U) \] (1.1)
where \((x,t) \in \mathbb{R} \times \mathbb{R}_+, 0 < \alpha \leq 1, D_t^\alpha u \) and \(D_x^\alpha u \) are modified Riemann-Liouville derivatives, \(a(t) \) and \(b(t) \) are bounded measurable or integrable functions on \(\mathbb{R}_+ \), \(W(t) \) is Gaussian white noise, i.e., \(W(t) = \dot{\beta}(t) \) and \(\beta(t) \) is a Brownian motion, \(R(u, D_t^\alpha u, D_x^{3\alpha} u) = -pu D_t^\alpha u - q D_x^{3\alpha} u \) is a functional of \(u, D_t^\alpha u \) and \(D_x^{3\alpha} u \) for some constants \(p, q \) and \(R^\circ \) is the Wick version of the functional \(R \). In [13], Xie discussed Eq. (1.1) with \(\alpha = 1 \) and gave exact solutions by using the homogeneous balance principle which was given by Wang in [11]. The homogeneous balance method has been widely applied to derive nonlinear transformations, exact solitary wave solutions, auto-Bäcklund transformations and similarity reductions of nonlinear partial differential equations (PDE) in mathematical physics. These subjects have been researched by many authors, such as Wang [11], M. L. Wang and Y. M. Wang [12], Fan [4, 5], etc. On the basis of the white noise functional theory [10], white noise functional solutions for nonlinear stochastic PDE have been studied by many authors, e.g., Xie [13, 14], Chen and Xie [2, 3], Ghany et. al. [6, 7, 8, 9], and so on.

In this paper, we present a new approach to study the nonlinear stochastic FPDE in Wick versions and with the modified Riemann-Liouville derivatives by using the fundamental basics of white noise analysis, such as Wick product, Hermite transform and Kondratiev spaces. By employing this approach and the homogeneous balance principle, we get white noise functional solutions, especially stochastic single soliton and multi-soliton solutions to the Wick-type stochastic fractional KdV equations (1.1).

2 Framework

We assume that the reader is familiar with the definition and general properties of the Kondratiev spaces of stochastic distributions \((S)_n^{n-1}(n = 1, 2, ...)\)(see [10] for more details). The Wick product \(X \odot Y \) of two elements \(X = \sum_{\mu} a_\mu H_\mu, Y = \sum_{\nu} b_\nu H_\nu \in (S)_n^{n-1} \) with \(a_\mu, b_\nu \in \mathbb{R}^n \) is defined by

\[
X \odot Y = \sum_{\mu,\nu} (a_\mu, b_\nu) H_{\mu+\nu}
\]

where \((a_\mu, b_\nu) \) is the usual inner product in \(\mathbb{R}^n \). For \(X = \sum_{\mu} a_\mu H_\mu \in (S)_n^{n-1} \) with \(a_\mu \in \mathbb{R}^n \), the Hermite transform of \(X \), denoted by \(\mathcal{H}(X) \) or \(\tilde{X}(z) \), is defined by

\[
\mathcal{H}(X) = \tilde{X}(z) = \sum_{\mu} a_\mu z^\mu \in \mathbb{C}^n \quad (\text{when convergent}),
\]

where \(z = (z_1, z_2, ...) \in \mathbb{C}^n \) (the set of all sequences of complex numbers) and \(z^\mu = z_1^{\mu_1}z_2^{\mu_2}...z_n^{\mu_n} \) for \(\mu = (\mu_1, \mu_2, ...) \in \mathcal{J} \). For \(X, Y \in (S)_n^{n-1} \) by the above definitions we have

\[
\tilde{X} \odot \tilde{Y}(z) = \tilde{X}(z) \cdot \tilde{Y}(z),
\]
for all } z \text{ such that } \tilde{X}(z) \text{ and } \tilde{Y}(z) \text{ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of } \mathbb{C}^n \text{ defined by }
(z_1^1, \ldots, z_n^1) \cdot (z_1^2, \ldots, z_n^2) = \sum_{i=1}^n z_1^1 z_i^2.

In what follows, by stochastic distribution process or \((\mathcal{S})_{n-1}\)-process we mean a measurable function \(u : \mathbb{R}^d \to (\mathcal{S})_{n-1}\). Also, the process \(u\) is called continuous, differentiable, \(C^1, C^k, \ldots\) etc., if the \((\mathcal{S})_{n-1}\)-valued function \(u\) has these properties, respectively. In order to investigate the stochastic FPDEs, we will give the following definitions and results.

**Definition 2.1.** Let \(u\) be a continuous \((\mathcal{S})_{n-1}\)-process, and let \(h > 0\) denote a constant discretization span. Define the forward operator \(FW_{x_k}(h)\), by

\[
FW_{x_k}(h)u(x) := u(x_1, \ldots, x_k + h, x_{k+1}, \ldots, x_d).
\]

Then for \(0 < \alpha \leq 1\), the \(\alpha\)-order fractional difference of \(u\) is defined by the expression

\[
\Delta^\alpha_{x_k} u(x) := (FW_{x_k}(h) - 1)^\alpha u(x) = \sum_{j=0}^\infty (-1)^j \binom{\alpha}{j} u(x_1, \ldots, x_k + (\alpha - j)h, x_{k+1}, \ldots, x_d),
\]

and its \(\alpha\)-order fractional derivative is given by

\[
D^\alpha_{x_k} u(x) = \lim_{h \downarrow 0} \frac{\Delta^\alpha_{x_k} u(x)}{h^\alpha}.
\]

provided the limit exists in \((\mathcal{S})_{n-1}\).

In terms of the Hermite transform the limit on the right-hand side of (2.3) exists if and only if there exists an element \(Y \in (\mathcal{S})_{n-1}\) such that \(\frac{1}{h^\alpha} \Delta^\alpha_{x_k} u(x, z) \to Y(z)\) point-wise bounded (uniformly) in \(K_\sigma(\delta)\) for some \(\sigma < \infty, \delta > 0\), where

\[
K_\sigma(\delta) = \{ z = (z_1, z_2, \ldots) \in \mathbb{C}^N : \sum_{\mu \neq 0} |z^\mu|^2 (2N)^{\sigma \mu} < \delta \}
\]

If this is the case, then \(Y\) is denoted by \(D^\alpha_{x_k} u(x)\).

Let us denote by \(L^1(a, b; (\mathcal{S})_{n-1})\) the space of all strongly integrable \((\mathcal{S})_{n-1}\)-processes on \([a, b]\), then for \(X \in L^1(a, b; (\mathcal{S})_{n-1})\) we can set the \(\alpha\)-order Riemann-Liouville fractional integral operator and the modified Riemann-Liouville fractional derivative as follows:

**Definition 2.2.** The \(\alpha\)-order Riemann-Liouville fractional integral operator of \(X\) is defined as

\[
J^\alpha X(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} X(\tau) d\tau,
\]

for \(\alpha > 0\), \(t \in [a, b]\) and \(J^0 X(t) := X(t)\).
Definition 2.3. The modified Riemann-Liouville fractional derivative of $X$ is defined as

$$D_x^\alpha X(t) = \begin{cases} \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{-\alpha-1}[X(\tau) - X(0)]d\tau, & \alpha < 0, \\ \frac{1}{\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha}[X(\tau) - X(0)]d\tau, & 0 < \alpha < 1, \\ [X^{(\alpha-n)}(t)]^{(n)}, & n \leq \alpha < n+1, \ n \in \mathbb{N} \end{cases}$$

(2.5)

When we apply Hermite transform to solve stochastic differential equations the following observation is important.

Assume that the $(S)_{-1}$-process $X(t, \omega)$ has an $\alpha$-order fractional derivative and $D_x^\alpha X(t, \omega) = F(t, \omega)$ in $(S)_{-1}$, this equivalent to saying that

$$\lim_{h \downarrow 0} \frac{\Delta_x^\alpha \tilde{X}(t, z)}{h^\alpha} = \tilde{F}(t, z)$$

(2.7)

uniformly for $z \in K_\sigma(\delta)$ for some $\sigma < \infty, \delta > 0$. For this it is clearly necessary that

$$D_x^\alpha \tilde{X}(t, z) = \tilde{F}(t, z) \quad \text{for all} \quad z \in K_\sigma(\delta),$$

(2.8)

but apparently not sufficient, because we also need that the pointwise convergence is bounded for $z \in K_\sigma(\delta)$. The following result is sufficient for our purposes.

Lemma 2.1. Suppose $X(t, \omega)$ and $F(t, \omega)$ are $(S)_{-1}$-processes such that

(i) $D_x^\alpha \tilde{X}(t, z) = \tilde{F}(t, z)$ for each $(t, z) \in (a, b) \times K_\sigma(\delta)$ and that

(ii) $\tilde{F}(t, z)$ is a bounded function for $(t, z) \in (a, b) \times K_\sigma(\delta)$ and continuous with respect to $t \in (a, b)$ for each $z \in K_\sigma(\delta)$.

Then $X(t, \omega)$ has an $\alpha$-order fractional derivative and for each $t \in (a, b)$

$$D_x^\alpha X(t, \omega) = F(t, \omega) \quad \text{in} \quad (S)_{-1}.$$  

(2.9)

Proof. According to the fractional counterpart of the mean value theorem, we have

$$\frac{1}{h^\alpha} \Delta_x^\alpha \tilde{X}(t, z) = \frac{\Gamma(1+\alpha)}{h^\alpha} (\tilde{X}(t+h, z) - \tilde{X}(t, z)) = \tilde{F}(t+\theta h, z),$$

(2.10)

for some $\theta \in [0, 1]$ and for each $z \in K_\sigma(\delta)$. So if the hypotheses (i), (ii) hold, then

$$\lim_{h \downarrow 0} \frac{\Delta_x^\alpha \tilde{X}(t, z)}{h^\alpha} = \tilde{F}(t, z)$$

(2.11)
Taking Hermite transform of (2.4) and using [10, Lemma 2.8.5], we get the following result

**Lemma 2.2.** Let $X(t)$ be an $(S)_{-1}$-process. Suppose there exist $\sigma < \infty, \delta > 0$ such that

$$\sup \{ \tilde{X}(t,z) : t \in [a,b], \ z \in K_\sigma(\delta) \} < \infty \quad (2.12)$$

and $\tilde{X}(t,z)$ is a continuous function with respect to $t \in [a,b]$ for each $z \in K_\sigma(\delta)$. Then the $\alpha$-order Riemann-Liouville fractional integral operator of $X(t)$ exists and

$$J^\alpha \tilde{X}(t)(z) = J^\alpha \tilde{X}(t,z), \quad \text{for } \alpha \geq 0, \ t \in [a,b], \ z \in K_\sigma(\delta). \quad (2.13)$$

In the case of higher order derivatives we have the following result

**Lemma 2.3.** Suppose there exist an open interval $I$, real numbers $\sigma, \delta$ and a function $u : I \times K_\sigma(\delta) \to \mathbb{C}$ such that

$$D^2_{2\alpha}x u(x,z) = \bar{F}(x,z), \quad \text{for } (x,z) \in I \times K_\sigma(\delta) \quad (2.14)$$

where $F(x) \in (S)_{-1}$ for all $x \in I$. Suppose $D^2_{2\alpha}x u$ is bounded for $(x,z) \in I \times K_\sigma(\delta)$ and continuous with respect to $x \in I$ for each $z \in K_\sigma(\delta)$. Then there exists $U(x) \in (S)_{-1}$ such that

$$D^2_{2\alpha}x U(x) = F(x), \quad \text{for } x \in I. \quad (2.15)$$

**Proof.** By the fractional counterpart of the mean value theorem again, we have

$$\frac{1}{h^{2\alpha}} \Delta^2_{2\alpha} x u(x,z) = \frac{\Gamma^2(1 + \alpha)}{h^{2\alpha}} (u(x + 2h,z) - 2u(x + h,z) + u(x,z)) = \bar{F}(x + \theta h,z) \quad (2.16)$$

for some $\theta \in [0,1]$ and for each $z \in K_\sigma(\delta)$. So if (2.14) and the assumptions on $D^2_{2\alpha}x u$ hold, then

$$\lim_{h \to 0} \frac{\Delta^2_{2\alpha} x u(t,z)}{h^{2\alpha}} = \bar{F}(x,z) \quad (2.17)$$

pointwise boundedly for $z \in K_\sigma(\delta)$. According to [10, Lemma 2.8.5], we can apply the inverse Hermite transform to Eq.(2.17) and get

$$D^2_{2\alpha}x U(x) = F(x) \quad \text{in } (S)_{-1} \quad \text{and for all } x \in I, \quad (2.18)$$

where $u(x,z) = \tilde{U}(x)(z)$ for all $(x,z) \in I \times K_\sigma(\delta)$. \hfill \qed

More generally, we can apply the argument of Lemma 2.1 repeatedly and get the following result
Theorem 2.4. Suppose $u(x, t, z)$ is a solution (in the usual strong, pointwise sense) of the equation

$$\tilde{\Omega}(x, t, D_t^\alpha, D_{x_1}^\alpha, \ldots, D_{x_d}^\alpha, u, z) = 0$$

(2.19)

for $(x, t)$ in some bounded open set $G \subset \mathbb{R}^d \times \mathbb{R}_+$, and for all $z \in K_\sigma(\delta)$, for some $\sigma, \delta$. Moreover, suppose that $u(x, t, z)$ and all its partial fractional derivatives, which are involved in (2.19), are (uniformly) bounded for $(x, t, z) \in G \times K_\sigma(\delta)$, continuous with respect to $(x, t) \in G$ for each $z \in K_\sigma(\delta)$ and analytic with respect to $z \in K_\sigma(\delta)$, for all $(x, t) \in G$. Then there exists $U(x, t) \in (S)_{-1}$ such that $u(x, t, z) = \tilde{U}(t, x)(z)$ for all $(t, x, z) \in G \times K_\sigma(\delta)$ and $U(x, t)$ solves (in the strong sense) the equation

$$\Omega^c(t, x, D_t^\alpha, D_{x_1}^\alpha, \ldots, D_{x_d}^\alpha, U, \omega) = 0 \quad \text{in} \quad (S)_{-1}.$$  

(2.20)

3 Soliton Solutions of Eq.(1.1)

In this section, we will give exact solutions of Eq.(1.1). Taking the Hermite transform of Eq.(1.1), we can get the equation

$$D_t^\alpha \tilde{U}(x, t, z) + [a(t) + p\tilde{W}(t, z)]\tilde{U}(x, t, z)D_x^\alpha \tilde{U}(x, t, z) + [b(t) + q\tilde{W}(t, z)]D_x^{3\alpha}\tilde{U}(x, t, z) = 0,$$

(3.1)

where the Hermite transform of $W(t)$ is defined by $\tilde{W}(t, z) = \sum_{k=1}^\infty z_k \int_0^t \eta_k(s) ds$ and $z \in (\mathbb{C}^n)_c$ is a vector parameter. We aim to solve Eq.(3.1).

For the sake of simplicity, we denote $u(x, t, z) = \tilde{U}(x, t, z)$, $A(t, z) = a(t) + p\tilde{W}(t, z)$ and $B(t, z) = b(t) + q\tilde{W}(t, z)$. Using the homogeneous balance principle, we can suppose the solution of Eq.(3.1) in the form

$$u(x, t, z) = D_x^{2\alpha}[\chi(\varphi(x, t, z))] + V(x, t, z) = \chi''(D_x^{2\alpha}\varphi)^2 + \chi'(D_x^{2\alpha}\varphi) + V(x, t, z),$$

(3.2)

where $\chi = \chi(\varphi)$ is a function of one argument only, $\chi' = d\chi/d\varphi, \chi^{(k)} = d^k\chi/d\varphi^k, \chi(\varphi)$ and $\varphi(x, t, z)$ are functions to be determined later and $V = V(x, t, z)$ is a given solution of Eq.(3.1) for any $z \in (\mathbb{C}^n)_c$ which may be a trivial one, a constant one, and so on. Using Eq.(3.2) we have

$$AuD_x^{2\alpha}u + BD_x^{2\alpha}u = [A\chi''\chi''' + B\chi^{(5)}]D_x^{5\alpha}\varphi + ..., $$

(3.3)

where the unwritten part in Eq.(3.3) is a polynomial of various fractional partial derivatives of $\varphi(x, t, z)$ (in spite of $\chi(\varphi)$ and its derivatives), the degree of which is lower than 5. Setting the coefficient of $D_x^{5\alpha}\varphi$ in Eq.(3.3) to zero yields an ordinary differential equation (ODE) for $\chi(\varphi)$

$$A\chi''\chi''' + B\chi^{(5)} = 0.$$

(3.4)
Since $\chi(\varphi)$ is a function of $\varphi$ only, then we can suppose that $A(t, z)$ and $B(t, z)$ are linearly dependant, that is

$$B(t, z) = \lambda A(t, z), \quad \text{for all } (t, z) \in \mathbb{R}_+ \times (\mathbb{C}^N)_c, \quad \lambda = \text{const} \neq 0$$  \hspace{1cm} (3.5)

Under condition (3.5), the ODE (3.4) admits a solution

$$\chi(\varphi) = 12\lambda \ln \varphi$$  \hspace{1cm} (3.6)

Substituting Eq.(3.6) into Eq.(3.2) yields

$$u(x, t, z) = 12\lambda D_x^{2\alpha}(\ln \varphi) + V(x, t, z).$$  \hspace{1cm} (3.7)

Now, we look for an equation satisfied by $\varphi(x, t, z)$ such that expression (3.7) is actually a solution of Eq.(3.1). Substituting Eq.(3.7) into the left-hand side of Eq.(3.1), we have

$$D_t^\alpha u + Au^\alpha u + BD_x^{3\alpha} u = 12\lambda D_x^{2\alpha}\left\{ \frac{1}{\varphi^2} \left[ \varphi(D_x^\alpha D_t^\alpha \varphi + \lambda A D_x^{4\alpha} \varphi + V A D_x^{2\alpha} \varphi) - (D_t^\alpha \varphi + \lambda A D_x^{3\alpha} \varphi + V A D_x^{\alpha} \varphi) D_x^\alpha \varphi + 3\lambda A (D_x^{2\alpha} \varphi)^2 - D_x^\alpha \varphi D_x^{3\alpha} \varphi \right] \right\}$$  \hspace{1cm} (3.8)

Therefore, expression (3.7) is a solution of Eq.(3.1) in condition that

$$\varphi \left( D_x^\alpha D_t^\alpha \varphi + \lambda A D_x^{4\alpha} \varphi + V A D_x^{2\alpha} \varphi \right) - (D_t^\alpha \varphi + \lambda A D_x^{3\alpha} \varphi + V A D_x^{\alpha} \varphi) D_x^\alpha \varphi + 3\lambda A (D_x^{2\alpha} \varphi)^2 - D_x^\alpha \varphi D_x^{3\alpha} \varphi = 0,$$  \hspace{1cm} (3.9)

for all $z \in (\mathbb{C}^N)_c$. Expression (3.7) in combination with Eq.(3.9) form an FBT of Eq.(3.1). The meaning of this FBT is that if $V = V(x, t, z)$ is a given solution of Eq.(3.1) and $\varphi = \varphi(x, t, z)$ is a solution of Eq.(3.9), then expression (3.7) is another solution of Eq.(3.1).

Now we use the FBT (3.7) and (3.9) to obtain the solitary wave solutions of Eq.(3.1). Since Eq.(3.9) is nonlinear, it is difficult to solve it in general, especially when $V = V(x, t, z)$ is a general function. However, if we take $V = V_0 = \text{const}$ as a solution of Eq.(3.1), then Eq.(3.9) becomes

$$\varphi \left( D_x^\alpha D_t^\alpha \varphi + \lambda A D_x^{4\alpha} \varphi + V_0 A D_x^{2\alpha} \varphi \right) - (D_t^\alpha \varphi + \lambda A D_x^{3\alpha} \varphi + V_0 A D_x^{\alpha} \varphi) D_x^\alpha \varphi + 3\lambda A (D_x^{2\alpha} \varphi)^2 - D_x^\alpha \varphi D_x^{3\alpha} \varphi = 0,$$  \hspace{1cm} (3.10)

which is easily to be solved. In fact, the first two parts of Eq.(3.10) involve the linear fractional operator $D_t^\alpha + \lambda A D_x^{3\alpha} + V_0 A D_x^\alpha$. With the help of modified fractional sub-equation method [9], the corresponding linear equation $D_t^\alpha \varphi + \lambda A D_x^{3\alpha} \varphi + V_0 A D_x^\alpha \varphi$ admits
a Mittag-Leffler function solution
\[ \varphi(x, t, z) = 1 + E_{\alpha}\{\xi(x, t, z)\}, \] (3.11)

where
\[ \xi(x, t, z) = s^\alpha \left[x - (\lambda s^{2\alpha} + V_0)^{\frac{1}{\alpha}} \int_0^t A(t, z) d\tau + x_0\right], \] (3.12)
s and \( x_0 \) are arbitrary parameters and \( E_{\alpha}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j+1)} \) is the Mittag-Leffler function.

The third part of Eq.(3.10) vanishes due to the homogeneity in the derivatives if we substitute (3.11) with (3.12) in it. So we see immediately that (3.11) with (3.12) is a solution of Eq.(3.10) for any \( s \) and \( x_0 \).

Substituting (3.11) with (3.12) into Eq.(3.7) gives the following single solitary wave solution of Eq.(3.1)
\[ u(x, t, z) = 12\lambda s^{2\alpha^2} \frac{E_{\alpha}\{\xi(x, t, z)\}}{(1 + E_{\alpha}\{\xi(x, t, z)\})^{\frac{1}{2}}} + V_0, \] (3.13)

where \( \xi(x, t, z) \) is expressed by Eq.(3.12).

By using Eqs.(3.12), (3.13) and the definition of \( \tilde{W} \), it is easy to prove that there exist a bounded open set \( G \subset \mathbb{R} \times \mathbb{R}_+ \) such that \( u(x, t, z) \) and \( D^3_x u(x, t, z) \) are uniformly bounded for \( (x, t, z) \in G \times K_{\sigma}(\delta) \) and continuous with respect to \( (x, t) \in G \) for all \( z \in K_{\sigma}(\delta) \) and analytic with respect to \( z \in K_{\sigma}(\delta) \) for all \( (x, t) \in G \). Theorem 2.4 implies that there exists \( U(x, t) \in \mathcal{S}_{-1} \) such that \( u(x, t, z) = \langle \mathcal{H} U(x, t) \rangle(z) \) for all \( (x, t, z) \in G \times K_{\sigma}(\delta) \) and that \( U(x, t) \) solves Eq.(1.1). From the above, we have that \( U(x, t) \) is the inverse Hermite transform of \( u(x, t, z) \). Hence, by Eqs.(3.12), (3.13) we have the following stochastic single soliton solution of Eq.(1.1)
\[ U(x, t) = 12\lambda s^{2\alpha^2} \frac{E^\circ\alpha\{\Xi(x, t)\}}{(1 + E^\circ\alpha\{\Xi(x, t)\})^{\frac{1}{2}}} + V_0, \] (3.14)

where
\[ \Xi(x, t) = s^\alpha \left[x - (\lambda s^{2\alpha} + V_0)^{\frac{1}{\alpha}} \int_0^t A(\tau, z) d\tau + x_0\right] \]
\[ = s^\alpha \left[x - (\lambda s^{2\alpha} + V_0)^{\frac{1}{\alpha}} \left(p\beta(t) + \int_0^t a(\tau) d\tau\right) + x_0\right] \] (3.15)

By using the relation \( E^\circ\alpha\{\beta(t)\} = E_{\alpha}\{\beta(t) - \frac{1}{4}t^2\} \) (see [10]) and Eqs.(3.14), (3.15), we have the solution of Eq.(1.1) in non-Wick version as follows
\[ U(x, t) = 12\lambda s^{2\alpha^2} \frac{E_{\alpha}\{\Xi^*(x, t)\}}{(1 + E_{\alpha}\{\Xi^*(x, t)\})^{\frac{1}{2}}} + V_0, \] (3.16)
where

$$\Xi^\alpha(x,t) = s^\alpha \left[ x - (\lambda s^{2\alpha} + V_0)^{\frac{1}{\alpha}} \left( p^\beta(t) - \frac{1}{2} pt^2 + \int_0^t a(\tau)d\tau \right) + x_0 \right]. \quad (3.17)$$

4 Conclusion

This paper is devoted to use white noise setting, in particular Wick product, Hermite transform and Kondratiev spaces to present a new approach to study the nonlinear stochastic fractional systems with the modified Riemann-Liouville derivatives. Then by using this approach and the homogeneous balance principle an FBT and white noise functional solutions for the Wick-type stochastic fractional KdV equations are showed. Obviously, the planner which we have proposed in this paper can be also applied to other nonlinear PDEs in mathematical physics such as KdV-Burgers, modified KdV-Burgers, Sawada-Kotera, Zhiber-Shabat and Benjamin-Bona-Mahony equations. Note that, if $\alpha = 1$, Eq.(1.1) is reduced to the Wick-type stochastic KdV equations. Hence, our results can be considered a generalization of the work due to Xie [13]. Moreover, there is a close mathematical connection between stochastic PDEs driven by Gaussian and Poissonian noise, at least for Wick-type equations. More precisely, there is a unitary map between the two spaces, such that one can obtain the solution of the Poissonian stochastic PDE simply by applying this map to the solution of the corresponding Gaussian stochastic PDE. A nice, concise account of this connection was given by Benth and Gjerde in [1]. Hence, we can get stochastic single soliton of Eq.(1.1).

References


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