Weak and Strong Convergence of Multi-Step Iterative Algorithm with Errors for $p + 1$ Asymptotically Nonexpansive Mappings

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Abstract

In this paper, we study multi-step iterative algorithm with errors for $p+1$ asymptotically nonexpansive mappings in uniformly convex Banach spaces. Also we have proved weak and strong convergence theorems for the mentioned algorithm. The results presented in this paper improve and extend the corresponding results of Khan and Fukhar-ud-din [6], Takahashi and Tamura [21], Boonchari and Saejung [1], Saluja [15], and many others.

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1 Introduction

Let $E$ be a real Banach space, $K$ be a nonempty subset of $E$. Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $F(T) = \{x \in E : Tx = x\}$ the set of fixed points of a mapping $T$. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. $T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$. And $T$ is said to be uniformly $L$-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$, $\|Tx - Ty\| \leq L \|x - y\|$. It is easy to see that every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian, but the converse is not true.

The class of asymptotically nonexpansive mappings which is an important generalization of that nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. They proved that, if $K$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $E$, then every asymptotically nonexpansive self-mapping of $K$ has a fixed point. Moreover, the set $F(T)$ of fixed points of $T$ is closed and convex. Since 1972, many authors have studied weak and strong convergence problem of the iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [4], [6], [7], [13], [17], [22] and references therein).

Asymptotically nonexpansive mappings have been widely and extensively studied by many authors in many aspects. One is to approximate a fixed point or a common fixed point of asymptotically nonexpansive mappings by means of an iteratively constructed sequence.

In recent years, Mann iterative scheme [11], Ishikawa iterative scheme [5] and Noor iterative scheme [22] have been studied extensively by many authors. In 1995, Liu [8] introduced iterative schemes with errors as follows:

\begin{align*}
x_1 &= x \in K, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tx_n + u_n, \quad (1)
\end{align*}

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{u_n\}$ a sequence in $E$ satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, is known as Mann iterative scheme with errors.
The sequence \( \{x_n\} \) defined by
\[
\begin{align*}
    x_1 &= x \in K, \\
    x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n, \\
    y_n &= (1 - \beta_n)x_n + \beta_n Tx_n + v_n,
\end{align*}
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \([0, 1]\), \(\{u_n\}\) and \(\{v_n\}\) are sequences in \(E\) satisfying \(\sum_{n=1}^{\infty} \|u_n\| < \infty\) and \(\sum_{n=1}^{\infty} \|v_n\| < \infty\), is known as Ishikawa iterative scheme with errors.

While it is clear that consideration of errors terms in iterative scheme is an important part of the theory, it is also clear that the iterative scheme with errors introduced by Liu [8], as in (1), (2) above, are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms in (1), (2) which say that they tend to zero as \(n\) tends to infinity are, therefore, unreasonable. Xu [23] introduced a more satisfactory error term in the following iterative schemes.

The sequence \( \{x_n\} \) defined by
\[
\begin{align*}
    x_1 &= x \in K, \\
    x_{n+1} &= \alpha_n Tx_n + \beta_n x_n + \gamma_n u_n,
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\} \) and \( \gamma_n \) are sequences in \([0, 1]\) such that \(\alpha_n + \beta_n + \gamma_n = 1\) and \(\{u_n\}\) is a bounded sequence in \(K\), is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if \(\gamma_n = 0\).

The sequence \( \{x_n\} \) defined by
\[
\begin{align*}
    x_1 &= x \in K, \\
    x_{n+1} &= \alpha_n Ty_n + \beta_n x_n + \gamma_n u_n, \\
    y_n &= \alpha'_n Tx_n + \beta'_n x_n + \gamma'_n v_n,
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\} \) ans \( \{\gamma'_n\} \) are sequences in \([0, 1]\) such that \(\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1\), \(\{u_n\}\) and \(\{v_n\}\) are bounded sequences in \(K\), is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme if \(\gamma_n = \gamma'_n = 0\). Chidume and Moore [2] and Takahashi and Tamura [21] studied the above schemes, respectively.

Many authors starting from Das and Debata [3] and including Khan and Takahashi [7], Shahzad and Udomene [19] and Takahashi and Tamura [21] have studied the two mappings case of iterative schemes for different types of mappings.

In 2005, Khan and Fukhar-ud-din [6] generalized iterative scheme (4) to
the one with errors as follows

\[ x_n = x \in K, \]
\[ y_n = \alpha_n' T x_n + \beta_n' x_n + \gamma_n' v_n, \]
\[ x_{n+1} = \alpha_n S y_n + \beta_n x_n + \gamma_n u_n, \quad n \geq 1 \]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \) are sequences in \([0, 1]\) satisfying \( \alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n' \) and \( \{u_n\}, \{v_n\} \) are bounded sequence in \( K \) with \( 0 < \delta \leq \alpha_n, \alpha_n' \leq 1 - \delta < 1 \).

Boonchari and Saenjung [1] generalized the scheme (5) to three nonexpansive mappings with errors as follows:

The sequence \( \{x_n\} \) defined by

\[ x_1 = x_0 \in K, \]
\[ y_n = \alpha_n' R x_n + \beta_n' T x_n + \gamma_n' v_n, \]
\[ x_{n+1} = \alpha_n R x_n + \beta_n S y_n + \gamma_n u_n, \quad n \geq 1 \]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\} \) are sequences in \([0, 1]\) satisfying \( \alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n' \) and \( \{u_n\}, \{v_n\} \) are bounded sequence in \( K \) with \( 0 < \delta \leq \beta_n, \beta_n' \leq 1 - \delta < 1 \).

Recently, Saluja [15] generalized the scheme (6) to four asymptotically nonexpansive mappings \( R, S, T, U \). The scheme is as follows:

\[ x_1 = x_0 \in K, \]
\[ y_n = \alpha_n'' R x_n + \beta_n'' T x_n + \gamma_n'' u_n, \]
\[ x_{n+1} = \alpha_n R x_n + \beta_n S y_n + \gamma_n u_n, \quad n \geq 1 \]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n''\}, \{\beta_n''\}, \{\gamma_n''\} \) are sequences in \([0, 1]\) satisfying \( \alpha_n + \beta_n + \gamma_n = \alpha_n'' + \beta_n'' + \gamma_n'' = 1 \) and \( \{u_n\}, \{v_n\}, \{w_n\} \) are bounded sequence in \( K \) with \( 0 < \delta \leq \beta_n, \beta_n'' \leq 1 - \delta < 1 \).

Inspired and motivated by [15] and [1], we extend the scheme (7) to the multi-step iteration scheme with errors for \( p + 1 \) asymptotically nonexpansive mappings \( R, T_i, i = 1, 2, \ldots, p \). The scheme is as follows:

\[ x_1 = x_0 \in K, \]
\[ x_{n+1} = \alpha_n R x_n + \beta_n T_i y_n^i + \gamma_n u_n^i, \]
\[ y_n^i = \alpha_n'' R x_n + \beta_n'' T_i y_{n+1}^i + \gamma_n'' u_{n+1}^i, \quad i = 1, 2, \ldots, p - 2 \]
\[ y_{n+1}^{p-1} = \alpha_n'' R x_n + \beta_n'' T_{p-1} y_{n+1}^{p-1} + \gamma_n'' u_{n+1}^{p-1}, \quad n \geq 1 \]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n''\}, \{\beta_n''\}, \{\gamma_n''\}, i = 1, 2, \ldots, p - 1 \) are sequences in \([0, 1]\) satisfying \( \alpha_n + \beta_n + \gamma_n = \alpha_n'' + \beta_n'' + \gamma_n'' = 1, i = 1, 2, \ldots, p - 1 \) and
Weak and strong convergence of multi-step iterative algorithm

\{u^1_n\}, \{u^2_n\}, \ldots, \{u^p_n\} are bounded sequence in \(K\) with \(0 < \delta \leq \beta_n, \beta^n_i \leq 1 - \delta < 1, i = 1, 2, \ldots, p - 1.\)

The purpose of this paper is to study multi-step iterative algorithm with errors (8) to approximate common fixed points of \(p + 1\) asymptotically nonexpansive mappings in uniformly convex Banach spaces. Our main results extend and improve the corresponding known results in the literatures.

2 Preliminary Notes

Let \(E\) be a Banach space and let \(K\) be a nonempty closed convex subset of \(E\). When \(\{x_n\}\) is a sequence in \(E\), we denote strong and weak convergence of \(\{x_n\}\) to \(x \in E\) by \(x_n \to x\) and \(x_n \rightharpoonup x\), respectively.

A Banach space \(E\) is said to satisfy Opial’s condition [12] if for any sequence \(\{x_n\}\) in \(E\), \(x_n \rightharpoonup x\) it follows that
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]
for all \(y \in E\) with \(y \neq x\). For every \(\varepsilon\) with \(0 \leq \varepsilon \leq 2\), we define the modulus \(\delta_E(\varepsilon)\) of convexity of \(E\) by
\[
\delta_E(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.
\]

A Banach space \(E\) is uniformly convex if for all \(\{x_n\}, \{x_n\} \subset \{z \in X : \|z\| = 1\} \) such that \(\frac{\|x_n - y_n\|}{2} \to 1\), we have \(\|x_n - y_n\| \to 0\). Next we state the following useful lemmas which will be essential for our main results:

Lemma 2.1 ([16]). Let \(E\) be a uniformly convex Banach space and \(0 < \alpha \leq t_n \leq \beta < 1\) for all \(n \in \mathbb{N}\). Suppose further that \(\{x_n\}\) and \(\{y_n\}\) are sequences of \(E\) such that
\[
\limsup_{n \to \infty} \|x_n\| \leq a, \limsup_{n \to \infty} \|y_n\| \leq a \quad \text{and} \quad \lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = a
\]
hold for some \(a \geq 0\), then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\).

Lemma 2.2 ([20]). Let \(\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty\) and \(\{r_n\}_{n=1}^\infty\) be sequences of non-negative numbers satisfying the inequality
\[
\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \forall n \geq 1.
\]
If \(\sum_{n=1}^\infty \beta_n < \infty\) and \(\sum_{n=1}^\infty r_n < \infty\), then \(\lim_{n \to \infty} \alpha_n\) exists. In particular, if \(\{\alpha_n\}_{n=1}^\infty\) has a subsequence which converges to zero, then \(\lim_{n \to \infty} \alpha_n = 0\).
3 Main Results

In this section, we shall prove the weak and strong convergence theorems of the iteration scheme (8) to a common fixed point of the asymptotically nonexpansive mappings $R, T_i, i = 1, 2, \ldots, p$.

We first establish the weak convergence theorem for the scheme (8).

**Theorem 3.1.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset of $E$ which satisfies Opial’s condition. Let $R, T_i : K \to K, i = 1, 2, \ldots, p$ be $p + 1$ asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ and such that $\sum_{n=1}^{\infty} (k_n^i - 1) < \infty$. Let $\{x_n\}$ be the sequence defined in (8), with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma_n^i < \infty, \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \gamma_n^i = 0, i = 1, 2, \ldots, p - 1$, converges weakly to a common fixed point of $R, T_i, i = 1, 2, \ldots, p$, where $F = F(R) \cap F(T_1) \cap \cdots \cap F(T_p) \neq \emptyset$.

**Proof.** Let $p^* \in F = F(R) \cap F(T_1) \cap \cdots \cap F(T_p)$. Since $R, T_i, i = 1, 2, \ldots, p$ are asymptotically nonexpansive mappings, from (8) we have

\[
\|y_n^{p-1} - p^*\| = \|\alpha_n^{p-1} R^n x_n + \beta_n^{p-1} T_p^n x_n + \gamma_n^{p-1} u_n^p - p^*\|
\leq \alpha_n^{p-1} \|R^n x_n - p^*\| + \beta_n^{p-1} \|T_p^n x_n - p^*\| + \gamma_n^{p-1} \|u_n^p - p^*\|
\leq \alpha_n^{p-1} k_n \|x_n - p^*\| + \beta_n^{p-1} k_n \|x_n - p^*\| + \gamma_n^{p-1} \|u_n^p - p^*\|
= (\alpha_n^{p-1} + \beta_n^{p-1}) k_n \|x_n - p^*\| + \gamma_n^{p-1} \|u_n^p - p^*\|
= (1 - \gamma_n^{p-1}) k_n \|x_n - p^*\| + \gamma_n^{p-1} \|u_n^p - p^*\|
\leq k_n \|x_n - p^*\| + A_n^1
\] (9)

where $A_n^1 = \gamma_n^{p-1} \|u_n^p - p^*\|$. Since $\sum_{n=1}^{\infty} \gamma_n^{p-1} < \infty$, it follows that $\sum_{n=1}^{\infty} A_n^1 < \infty$. 
Again from (8) and (9), we obtain

\[
\|y_n - p\| = \|\alpha_n R_n x_n + \beta_n T_n y_n + \gamma_n u_n - p\| \\
\leq \alpha_n \|R_n x_n - p\| + \beta_n \|T_n y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq k_n \|x_n - p\| + A_n^2
\]

(10)

where \(A_n^2 = \beta_n \|k_n A_n^1 + \gamma_n \|u_n - p\|\). Since \(\sum_{n=1}^{\infty} A_n^1 < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n \|u_n - p\| < \infty\), it follows that \(\sum_{n=1}^{\infty} A_n^2 < \infty\).

Similarly, we can prove that

\[
\|y_n - p\| = \|\alpha_n R_n x_n + \beta_n T_n^2 y_n + \gamma_n u_n - p\|
\]

\[
\leq \alpha_n \|R_n x_n - p\| + \beta_n \|T_n^2 y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\|
\]

\[
\leq k_n \|x_n - p\| + A_n^2
\]

(11)

where \(A_n^2 = \beta_n \|k_n A_n^{p-2} + \gamma_n \|u_n - p\|\). Since \(\sum_{n=1}^{\infty} A_n^{p-2} < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\), it follows that \(\sum_{n=1}^{\infty} A_n^{p-1} < \infty\).
From (8) and (11), we obtain
\[
\|x_{n+1} - p^*\| = \|\alpha_n R^n x_n + \beta_n T^n_1 y_n^1 + \gamma_n u_n^1 - p^*\|
\leq \alpha_n \|R^n x_n - p^*\| + \beta_n \|T^n_1 y_n^1 - p^*\| + \gamma_n \|u_n^1 - p^*\|
\leq \alpha_n k_n \|x_n - p^*\| + \beta_n k_n \|y_n^1 - p^*\| + \gamma_n \|u_n^1 - p^*\|
\leq \alpha_n k_n \|x_n - p^*\| + \beta_n k_n (k_n - 1) \|x_n - p^*\| + \gamma_n \|u_n^1 - p^*\|
\leq \alpha_n + \beta_n k_n \|x_n - p^*\| + \beta_n k_n (k_n - 1) \|x_n - p^*\| + \gamma_n \|u_n^1 - p^*\|
= (1 - \gamma_n) k_n \|x_n - p^*\| + \left(\beta_n k_n (k_n - 1) + \gamma_n \|u_n^1 - p^*\|\right)
\leq k_n \|x_n - p^*\| + A_n^p
\leq (1 + (k_n^p - 1)) \|x_n - p^*\| + A_n^p
\tag{12}
\]
where \(A_n^p = \beta_n k_n (k_n - 1) + \gamma_n \|u_n^1 - p^*\|\). Since \(\sum_{n=1}^{\infty} A_n^{p-1} < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\), it follows that \(\sum_{n=1}^{\infty} A_n^p < \infty\) and by assumption of the theorem \(\sum_{n=1}^{\infty} (k_n^p - 1) < \infty\). Hence, by Lemma 2.2, it follows that \(\lim_{n \to \infty} \|x_n - p\|\) exists, and so for \(n \geq 1\), the sequence \(\{x_n\}\) is bounded on \(K\).

Now, we show that \(\{x_n\}\) converges weakly to a common fixed point of \(R, T_i, i = 1, 2, \ldots, p\). By the reflexivity of \(E\) and the boundedness of \(\{x_n\}\), the sequence \(\{x_n\}\) contains a subsequence which converges weakly to the point in \(K\). Let \(\{x_{n_k}\}\) converges weakly to \(p^*\) and \(q^*\), respectively. We will show that \(p^* = q^*\). Suppose that \(E\) satisfies Opial’s condition and that \(p^* \neq q^*\) are in the weak limit set of the sequence \(\{x_n\}\). Then \(x_{n_k} \to p^*\) and \(x_{m_j} \to q^*\), respectively. Since \(\lim_{n \to \infty} \|x_n - p^*\|\) exists for any \(p^* \in \mathcal{F} = F(R) \cap F(T_1) \cap \cdots \cap F(T_p)\), by Opial’s condition we have that
\[
\lim_{n \to \infty} \|x_n - p^*\| = \lim_{k \to \infty} \|x_{n_k} - p^*\|
< \lim_{k \to \infty} \|x_{n_k} - q^*\| = \lim_{n \to \infty} \|x_n - q^*\| = \lim_{j \to \infty} \|x_{m_j} - q^*\|
< \lim_{j \to \infty} \|x_{m_j} - p^*\| = \lim_{n \to \infty} \|x_n - p^*\|.
\]
This is a contradiction. Hence \(\{x_n\}\) converges weakly to a common fixed point of \(\mathcal{F} = F(R) \cap F(T_1) \cap \cdots \cap F(T_p)\). \(\square\)

We recall that a mappings \(T : K \to K\) where \(K\) is a subset of \(E\), is said to satisfy condition (A) [18] if there exists a nondecreasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0, f(r) > 0\) for all \(r \in (0, \infty)\) such that
\[
\|x - T x\| \geq f(d(x, \mathcal{F}))
\]
for all $x \in K$ where $d(x, F) = \inf \{\|x - p\| : p \in F\}$.

Four mappings $R, S, T, U : K \to K$ where $K$ is a subset of $E$, is said to satisfy condition $(GA)$ [15] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$a_1\|x - Rx\| + a_2\|x - Sx\| + a_3\|x - Tx\| + a_4\|x - Ux\| \geq f(d(x, F))$$

for all $x \in K$ where $d(x, F) = \inf \{\|x - p\| : p \in F\}$, and $a_1, a_2, a_3$ and $a_4$ are four nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$.

We modify the condition $(GA)$ for $p + 1$ mappings $R, T_i : K \to K, i = 1, 2, \ldots, p$ as follows.

The mappings $R, T_i : K \to K, i = 1, 2, \ldots, p$ where $K$ is a subset of $E$, are said to satisfy condition $(JSA)$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$a_1\|x - Rx\| + a_2\|x - T_1x\| + a_3\|x - T_2x\| + \cdots + a_{p+1}\|x - T_{p}x\| \geq f(d(x, F))$$

for all $x \in K$ where $d(x, F) = \inf \{\|x - p\| : p \in F\}$, and $a_i, i = 1, 2, \ldots, p + 1$ are nonnegative real numbers such that $\sum_{i=1}^{p+1} a_i = 1$

**Remark 3.2.** Condition $(JSA)$ reduces to

- **condition $(GA)$** when $p = 3$ and
- **condition $(A)$** when $R = T_i, i = 1, 2, \ldots, p$.

**Lemma 3.3.** Let $E$ be a uniformly convex Banach space and $K$ be a nonempty closed convex subset of $E$. Let $R, T_i : K \to K, i = 1, 2, \ldots, p$ be $p+1$ asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ and such that $\sum_{n=1}^{\infty} (k_n^i - 1) < \infty$. Let $\{x_n\}$ be the sequence defined in (8), with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^i < \infty$, $\lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \gamma_n^i = 0, i = 1, 2, \ldots, p - 1$ and $0 < t_1 \leq \beta_n, \beta_n^i \leq t_2 < 1, i = 1, 2, \ldots, p - 1$ for some $t_1, t_2 \in (0, 1)$. If $F = F(R) \cap F(T_1) \cap F(T_2) \cap \cdots \cap F(T_p) \neq \emptyset$ and

$$\|x - T_1y\| \leq \|Rx - T_1y\|, \quad \forall x, y \in K, \quad (13)$$

Then

$$\lim_{n \to \infty} \|Rx_n - x_n\| = \lim_{n \to \infty} \|T_1x_n - x_n\| = \lim_{n \to \infty} \|T_2x_n - x_n\| = \cdots = \lim_{n \to \infty} \|T_{p-1}x_n - x_n\| = \lim_{n \to \infty} \|T_px_n - x_n\| = 0,$$

for all $p^* \in F$. 
Proof. From Theorem 3.1 we know that \( \lim_{n \to \infty} \|x_n - p^*\| \) exists.

Let \( \lim_{n \to \infty} \|x_n - p^*\| = r \). Then if \( r = 0 \), we are done. Suppose that \( r > 0 \). Again from Theorem 3.1 we obtain

\[
\|x_{n+1} - p^*\| \leq k_n^p \|x_n - p^*\| + A_n^p \tag{14}
\]

where \( A_n^p = \beta_n k_n A_n^{p-1} + \gamma_n \|u_n^1 - p^*\| \) such that \( \sum_{n=1}^{\infty} A_n^p < \infty \). And

\[
\|y_n^i - p^*\| \leq k_n^{p-i} \|x_n - p^*\| + A_n^{p-i}, \quad i = 1, 2, \ldots, p - 1, \tag{15}
\]

where \( A_n^{p-i} = \beta_n k_n A_n^{p-i-1} + \gamma_n \|u_n^i - p^*\| \) such that \( \sum_{n=1}^{\infty} A_n^{p-i} < \infty \).

We first show that \( \lim_{n \to \infty} \|T_1 y_n^1 - x_n\| = 0 \). We observe that \( \{u_n^1 - R^n x_n - p^*\} \) is a bounded sequence, so

\[
\lim_{n \to \infty} \gamma_n \|u_n^1 - R x_n - p^*\| = 0. \tag{16}
\]

From (15) we have

\[
\|y_n^1 - p^*\| \leq k_n^{p-1} \|x_n - p^*\| + A_n^{p-1}, \quad n \geq 1
\]

where \( A_n^{p-1} = \beta_n k_n A_n^{p-2} + \gamma_n \|u_n^1 - p^*\| \) such that \( \sum_{n=1}^{\infty} A_n^{p-1} < \infty \).

Taking \( n \to \infty \) on both sides, we obtain

\[
\limsup_{n \to \infty} \|y_n^1 - p^*\| \leq \limsup_{n \to \infty} \left( k_n^{p-1} \|x_n - p^*\| + A_n^{p-1} \right) \\
= \limsup_{n \to \infty} \|x_n - p^*\| = r. \tag{17}
\]

Note that

\[
\limsup_{n \to \infty} \|T_1^n y_n^1 - p^*\| \leq \limsup_{n \to \infty} k_n \|y_n^1 - p^*\| = r. \tag{18}
\]

Also,

\[
\limsup_{n \to \infty} \|R^n x_n - p^*\| \leq \limsup_{n \to \infty} k_n \|x_n - p^*\| = r. \tag{19}
\]

From (8) and (16) we have

\[
r = \lim_{n \to \infty} \|x_{n+1} - p^*\| \\
= \lim_{n \to \infty} \|\alpha_n R^n x_n + \beta_n T_1 T_1^n y_n^1 + \gamma_n u_n^1 - p^*\| \\
= \lim_{n \to \infty} \|(1 - \beta_n) R^n x_n + \beta_n T_1 y_n^1 + \gamma_n u_n^1 - \gamma_n R^n x_n - p^*\| \\
= \lim_{n \to \infty} \|(1 - \beta_n)(R^n x_n - p^*) - \gamma_n p^* + \beta_n(T_1 y_n^1 - p^*) + \gamma_n (u_n^1 - R^n x_n - p^*)\| \\
= \lim_{n \to \infty} \|(1 - \beta_n)(R^n x_n - p^*) + \beta_n(T_1 y_n^1 - p^*)\| \tag{20}
\]
From (18)-(20) and using Lemma 2.1 we have
\[
\lim_{n \to \infty} \| R^n x_n - T_1^n y_n \| = 0. 
\] (21)

Using (13), it follows then that
\[
\| R^n x_n - x_n \| \leq \| R^n x_n - T_1^n y_n \| + \| T_1^n y_n - x_n \|
\leq 2 \| R^n x_n - T_1^n y_n \| \to 0 \text{ as } n \to \infty, 
\] (22)

from (21) and (22), implies that
\[
\| T_1^n y_n - x_n \| \leq \| T_1^n y_n - R^n x_n \| + \| R^n x_n - x_n \|
\to 0 \text{ as } n \to \infty. 
\] (23)

We observe that, for each \( n \geq 1 \),
\[
\| x_n - p^* \| \leq \| x_n - T_1^n y_n \| + \| T_1^n y_n - p^* \|
\leq \| x_n - T_1^n y_n \| + k_n \| y_n - p^* \|, 
\] (24)

using (23), we obtain
\[
\begin{align*}
 r &= \lim_{n \to \infty} \| x_n - p^* \| \leq \liminf_{n \to \infty} \| y_n^1 - p^* \|.
\end{align*}
\]

This together with (17) gives
\[
\lim_{n \to \infty} \| y_n^1 - p^* \| = r. 
\] (25)

Next, we will show that \( \lim_{n \to \infty} \| T_2^n y_n^2 - x_n \| = 0 \).

We observe that \( \{ u_n^2 - R^n x_n - p^* \} \) is a bounded sequence, so
\[
\lim_{n \to \infty} \gamma_n^1 \| u_n^2 - R^n x_n - p^* \| = 0. 
\] (26)

Now from (15) we have
\[
\| y_n^2 - p^* \| \leq k_n^{p-2} \| x_n - p^* \| + A_n^{p-2}, \quad n \geq 1
\]
where \( A_n^{p-2} = \beta_n^1 k_n A_n^{p-3} + \gamma_n^1 \| u_n^2 - p^* \| \) such that \( \sum_{n=1}^{\infty} A_n^{p-2} < \infty \).

Taking \( n \to \infty \) on both sides, we obtain
\[
\begin{align*}
\limsup_{n \to \infty} \| y_n^2 - p^* \| &\leq \limsup_{n \to \infty} (k_n^{p-2} \| x_n - p^* \| + A_n^{p-2}) \\
&= \limsup_{n \to \infty} \| x_n - p^* \| = r.
\end{align*}
\] (27)
Note that
\[
\limsup_{n \to \infty} \|T_n y_n^2 - p^*\| \leq \limsup_{n \to \infty} k_n \|y_n^2 - p^*\| = r. \tag{28}
\]

Consider, from (25), (8) and (26) we have
\[
r = \lim_{n \to \infty} \|y_n^1 - p^*\|
= \lim_{n \to \infty} \|\alpha_n^1 R^n x_n + \beta_n^1 T_n y_n^2 + \gamma_n^1 u_n - p^*\|
= \lim_{n \to \infty} \|(1 - \beta_n^1)R^n x_n + \beta_n^1 T_n y_n^2 + \gamma_n^1 u_n - \gamma_n^1 R^n x_n - p^*\|
= \lim_{n \to \infty} \|(1 - \beta_n^1)(R^n x_n - p^*) - \gamma_n^1 p^* + \beta_n^1(T_n y_n^2 - p^*) + \gamma_n^1(u_n - R^n x_n - p^*)\|
= \lim_{n \to \infty} \|(1 - \beta_n^1)(R^n x_n - p^*) + \beta_n^1(T_n y_n^2 - p^*)\| \tag{29}
\]

From (19), (28), (29) and using Lemma 2.1 we have
\[
\lim_{n \to \infty} \|R_n x_n - T_n y_n^2\| = 0. \tag{30}
\]

Using (30) and (22), it follows then that
\[
\|T_n y_n^2 - x_n\| \leq \|T_n y_n^2 - R_n x_n\| + \|R_n x_n - x_n\|
\to 0 \text{ as } n \to \infty, \tag{31}
\]

Again, we observe that for each \(n \geq 1\),
\[
\|x_n - p^*\| \leq \|x_n - T_n y_n^2\| + \|T_n y_n^2 - p^*\|
\leq \|x_n - T_n y_n^2\| + k_n \|y_n^2 - p^*\|, \tag{32}
\]

using (31), we obtain
\[
r = \lim_{n \to \infty} \|x_n - p^*\| \leq \liminf_{n \to \infty} \|y_n^2 - p^*\|.
\]

This together with (27) gives
\[
\lim_{n \to \infty} \|y_n^2 - p^*\| = r. \tag{33}
\]

Similarly, we can show that
\[
\begin{align*}
\lim_{n \to \infty} \|T_n y_n^i - x_n\| &= 0 \quad \text{and} \\
\lim_{n \to \infty} \|y_n^i - p^*\| &= r; \quad \forall i = 1, 2, \ldots, p - 1.
\end{align*}
\tag{34}
\]
We note that \( \{u_n^p - R_n^p x_n - p^*\} \) is a bounded sequence, so
\[
\lim_{n \to \infty} \gamma_n^{p-1} \|u_n^p - R_n^p x_n - p^*\| = 0.
\] (35)

Also,
\[
\limsup_{n \to \infty} \|T_n^p x_n - p^*\| \leq \limsup_{n \to \infty} k_n \|x_n - p^*\| = r.
\] (36)

From (34), (8) and (35) we have
\[
r = \lim_{n \to \infty} \|y_n^{p-1} - p^*\|
= \lim_{n \to \infty} \|\alpha_n^{p-1} R_n^p x_n + \beta_n^{p-1} T_n^p x_n + \gamma_n^{p-1} u_n^p - p^*\|
= \lim_{n \to \infty} \|(1 - \beta_n^{p-1}) R_n^p x_n + \beta_n^{p-1} T_n^p x_n + \gamma_n^{p-1} u_n^p - \gamma_n^{p-1} R_n^p x_n - p^*\|
= \lim_{n \to \infty} \|(1 - \beta_n^{p-1}) R_n^p x_n - p^* + \beta_n^{p-1} (T_n^p x_n - p^*)
+ \gamma_n^{p-1} (u_n^p - R_n^p x_n - p^*)\|
= \lim_{n \to \infty} \|(1 - \beta_n^{p-1}) R_n^p x_n - p^* + \beta_n^{p-1} (T_n^p x_n - p^*)\|.
\] (37)

From (36), (19), (37) and using Lemma 2.1 we have
\[
\lim_{n \to \infty} \|R_n^p x_n - T_n^p x_n\| = 0.
\] (38)

From (38) and (22), it follows that
\[
\|T_n^p x_n - x_n\| \leq \|T_n^p x_n - R_n^p x_n\| + \|R_n^p x_n - x_n\|
\to 0 \text{ as } n \to \infty.
\] (39)

Consequently, we have
\[
\|x_n - T_1^p x_n\| \leq \|x_n - T_1^p y_n^1\| + \|T_1^p y_n^1 - T_n^p x_n\|
\leq \|x_n - T_1^p y_n^1\| + k_n \|y_n^1 - x_n\|
= \|x_n - T_1^p y_n^1\| + k_n (\alpha_n^1 R_n^p x_n + \beta_n^1 T_n^p y_n^2 + \gamma_n^1 u_n^2) - x_n\|
\leq \|x_n - T_1^p y_n^1\| + k_n \alpha_n^1 \|R_n^p x_n - x_n\| + k_n \beta_n^1 \|T_n^p y_n^2 - x_n\|
+ k_n \gamma_n^1 \|u_n^2 - x_n\|.
\] (40)

Using (23), (22) and (31), we have
\[
\lim_{n \to \infty} \|x_n - T_1^p x_n\| = 0.
\] (41)

Again, we have
\[
\|x_n - T_2^p x_n\| \leq \|x_n - T_2^p y_n^2\| + \|T_2^p y_n^2 - T_n^p x_n\|
\leq \|x_n - T_2^p y_n^2\| + k_n \|y_n^2 - x_n\|
= \|x_n - T_2^p y_n^2\| + k_n (\alpha_n^2 R_n^p x_n + \beta_n^2 T_n^p y_n^3 + \gamma_n^2 u_n^3) - x_n\|
\leq \|x_n - T_2^p y_n^2\| + k_n \alpha_n^2 \|R_n^p x_n - x_n\| + k_n \beta_n^2 \|T_n^p y_n^3 - x_n\|
+ k_n \gamma_n^2 \|u_n^3 - x_n\|.
\] (42)
Using (34) and (22), we have
\[
\lim_{n \to \infty} \| x_n - T^1_n x_n \| = 0. \tag{43}
\]

By continuing similar method as above, we obtain that,
\[
\lim_{n \to \infty} \| R^n x_n - x_n \| = \lim_{n \to \infty} \| T^1_n x_n - x_n \| = \lim_{n \to \infty} \| T^2_n x_n - x_n \| = \ldots = \lim_{n \to \infty} \| T^p_n x_n - x_n \| = \lim_{n \to \infty} \| T^p_n \| x_n - x_n \| = 0.
\]

Again note that
\[
\| x_{n+1} - x_n \| \leq \alpha_n \| R^n x_n - x_n \| + \beta_n \| T^1_n y^1_n - x_n \| + \gamma_n \| u^1_n - x_n \|,
\]
using (22) and (23), we have
\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \tag{45}
\]

And
\[
\| y^1_n - x_n \| \leq \alpha^1_n \| R^n x_n - x_n \| + \beta^1_n \| T^2_n y^2_n - x_n \| + \gamma^1_n \| u^2_n - x_n \|,
\]
using (22) and (31), we have
\[
\lim_{n \to \infty} \| y^1_n - x_n \| = 0. \tag{47}
\]

Similarly, for \( i = 2, 3, \ldots, p - 2 \),
\[
\| y^i_n - x_n \| \leq \alpha^i_n \| R^n x_n - x_n \| + \beta^i_n \| T^i_{n+1} y^i_{n+1} - x_n \| + \gamma^i_n \| u^{i+1}_n - x_n \|,
\]
using (22) and (34), we have
\[
\lim_{n \to \infty} \| y^i_n - x_n \| = 0. \tag{49}
\]

Now, we have
\[
\| x_n - T_1 x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T_1^{n+1} x_{n+1} \|
+ \| T_1^{n+1} x_{n+1} - T_1^{n+1} x_n \| + \| T_1^{n+1} x_n - T_1 x_n \|. \tag{50}
\]

Since \( T_1 \) is uniformly \( L \)-Lipschitzian, we obtain that
\[
\| x_n - T_1 x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T_1^{n+1} x_{n+1} \|
+ L \| x_{n+1} - x_n \| + L \| T_1^{n} x_n - x_n \|. \tag{51}
\]

Using (41) and (44), we obtain that
\[
\lim_{n \to \infty} \| x_n - T_1 x_n \| = 0. \tag{52}
\]
Similarly, we can prove that
\[
\lim_{n \to \infty} \|Rx_n - x_n\| = \lim_{n \to \infty} \|T_2x_n - x_n\| = \lim_{n \to \infty} \|T_3x_n - x_n\| = \cdots = \lim_{n \to \infty} \|T_{p-1}x_n - x_n\| = \lim_{n \to \infty} \|T_px_n - x_n\| = 0.
\]
This completes our proof.

**Theorem 3.4.** Let \( E \) be a uniformly convex Banach space and \( K, \{x_n\} \) be taken as in Lemma (3.3). Let \( R, T_i : K \to K, i = 1, 2, \ldots, p \) be \( p + 1 \) asymptotically nonexpansive mappings satisfying condition (JSA). If \( \mathcal{F} = F(R) \cap F(T_1) \cap F(T_2) \cap \cdots \cap F(T_p) \neq \emptyset \), \( 0 < t_1 \leq \beta_n, \beta_n^i \leq t_2 < 1, i = 1, 2, \ldots, p-1 \) for some \( t_1, t_2 \in (0, 1) \) and \( R, T_1 \) satisfy the condition (13), then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( R, T_i, i = 1, 2, \ldots, p \).

**Proof.** By Theorem 3.1, we know that \( \lim_{n \to \infty} \|x_n - p^*\| \) exists for all \( p^* \in \mathcal{F} \). Let \( \lim_{n \to \infty} \|x_n - p^*\| = r \) for some \( r \geq 0 \). If \( r = 0 \), we are done. Suppose that \( r > 0 \). By Lemma 3.3 we know that
\[
\lim_{n \to \infty} \|Rx_n - x_n\| = \lim_{n \to \infty} \|T_1x_n - x_n\| = \cdots = \lim_{n \to \infty} \|T_px_n - x_n\| = 0
\]
And from Theorem 3.1 we obtain
\[
\|x_{n+1} - p^*\| \leq k_n^p \|x_n - p^*\| + A_n^p \\
\leq (1 + C_n) \|x_n - p^*\| + A_n^p
\]
where \( C_n = k_n^p - 1, A_n^p = \beta_n k_n A_n^{p-1} + \gamma_n \|u_n^1 - p^*\| \) with \( \sum_{n=1}^{\infty} A_n^p < \infty \) and \( \sum_{n=1}^{\infty} C_n < \infty \). This implies that
\[
d(x_{n+1}, \mathcal{F}) \leq (1 + C_n)d(x_n, \mathcal{F}) + A_n^p.
\]
Hence, by Lemma 2.2, \( d(x_n, \mathcal{F}) \) exists. By condition (JSA) and Lemma 3.3, we obtain that
\[
\lim_{n \to \infty} f(d(x_n, \mathcal{F})) = 0.
\]
Since \( f \) is nondecreasing function and \( f(0) = 0 \), therefore \( \lim_{n \to \infty} d(x_n, \mathcal{F}) = 0 \).

We now show that \( \{x_n\} \) is Cauchy sequence in \( E \). In fact, \( \sum_{n=1}^{\infty} C_n < \infty \), and \( x > 0, 1 + x \leq e^x, \) for all \( x > 0 \). From (53), for any \( m, n \geq 1 \) and \( p^* \in \mathcal{F}, \)
we have
\[
\|x_{n+m} - p^*\| \leq (1 + C_{n+m-1})\|x_{n+m-1} - p^*\| + A_{n+m-1}^p \\
\leq (1 + C_{n+m-1})(1 + C_{n+m-2})\|x_{n+m-2} - p^*\| + (1 + C_{n+m-1}) (A_{n+m-1}^p + A_{n+m-2}^p) \\
\leq \exp(C_{n+m-1} + C_{n+m-2})\|x_{n+m-2} - p^*\| + \exp(C_{n+m-1}) (A_{n+m-1}^p + A_{n+m-2}^p) \\
\vdots \\
\leq \exp(\sum_{i=n}^{n+m-1} C_i)\|x_n - p^*\| + \exp(\sum_{i=n}^{n+m-1} C_i) \sum_{i=n}^{n+m-1} A_i^p \\
\leq M\|x_n - p^*\| + M \left( \sum_{i=n}^{n+m-1} A_i^p \right) \\
(54)
\]
where \( M = \exp(\sum_{i=n}^{\infty} C_i) < \infty \).

Since \( \lim_{n \to \infty} d(x_n, \mathcal{F}) = 0 \). Let \( \varepsilon > 0 \) we choose a positive integer \( N_0 \), such that for all \( n \geq N_0 \),
\[
d(x_n, \mathcal{F}) < \frac{\varepsilon}{6M}; \quad \sum_{i=n_0}^{\infty} A_i^p < \frac{\varepsilon}{3M}.
\]
We choose \( p_0 \in \mathcal{F} \) such that
\[
\|x_{N_0} - p_0\| < \frac{\varepsilon}{6M}.
\]
Hence, for all \( n \geq N_0 \) and \( m \geq 1 \) and from (54), we have
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_0\| + \|x_n - p_0\| \\
\leq M\|x_{n_0} - p_0\| + M \left( \sum_{i=n}^{n+m-1} A_i^p \right) + M\|x_{n_0} - p_0\| + M \left( \sum_{i=n_0}^{n-1} A_i^p \right) \\
\leq 2M\|x_{n_0} - p_0\| + M \left( \sum_{i=n}^{\infty} A_i^p \right) + M \left( \sum_{i=n}^{\infty} A_i^p \right) \\
\leq 2M \frac{\varepsilon}{6M} + M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3M} = \varepsilon.
\]

Therefore \( \{x_n\} \) is a Cauchy sequence in \( E \). Since \( E \) is a Banach space, which is complete, implies that \( \{x_n\} \) is convergent. Assume that \( x_n \to p^* \in E \). Since \( K \) is closed and \( \{x_n\} \) is a sequence in \( K \) converging to \( p^* \), we conclude that \( p^* \in K \). We note that \( \mathcal{F} = F(R) \cap F(T_1) \cap F(T_2) \cap \cdots \cap F(T_p) \) is closed. Now, \( \lim_{n \to \infty} d(x_n, \mathcal{F}) = 0 \) and \( x_n \to p^* \) as \( n \to \infty \), the continuity of \( d(x, \mathcal{F}) \) implies that \( d(p^*, \mathcal{F}) = 0 \). Then \( p^* \in \mathcal{F} \). This completes our proof. \( \square \)
Corollary 3.5 ([15, Theorem 3.5]). Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$, and $\{x_n\}$ be defined as in (7). Let $R, S, T, U : K \to K$ be four asymptotically nonexpansive mappings satisfying certain condition (GA). If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset, 0 < t_1 \leq \beta_n, \beta_n' \leq t_2 < 1$ for some $t_1, t_2 \in (0, 1)$, then $\{x_n\}$ converges strongly to a common fixed point of the mappings $R, S, T$ and $U$.

Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. A mapping $T : K \to K$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in $K$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{n \to \infty} x_{n_j} = x \in K$. We are now ready for the next main result.

Theorem 3.6. Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $R, T_i : K \to K, i = 1, 2, \ldots, p$ be asymptotically nonexpansive mappings satisfying condition (JSA). If $\mathcal{F} = F(R) \cap F(T_1) \cap \cdots \cap F(T_p) \neq \emptyset, 0 < \alpha \leq b_n, b_n^i \leq \beta < 1$, for some $\alpha, \beta \in (0, 1)$ and $R, T_1$ satisfy condition (13). Suppose one of the mappings in $\{R, T_1, T_2, \ldots, T_p\}$ is semi-compact then $\{x_n\}$ defined by (8) converges strongly to a common fixed point of the mappings $R, T_1, T_2, \ldots, T_{p-1}, T_p$.

Proof. Suppose $R$ is semi-compact. By Lemma 3.3, we have

$$\lim_{n \to \infty} \|x_n - Rx_n\| = 0.$$ 

So there exists a subsequence $x_{n_j}$ of $\{x_n\}$ such that $\lim_{j \to \infty} x_{n_j} = p^* \in K$. Now Lemma 3.3 implies that $\lim_{n \to \infty} \|x_{n_j} - Rx_{n_j}\| = 0$ and $\lim_{n \to \infty} \|x_{n_j} - T_i x_{n_j}\| = 0, i = 1, 2, \ldots, p$ and so $\|p^* - Rp^*\| = 0$ and $\|p^* - T_i p^*\| = 0, i = 1, 2, \ldots, p$. This implies that $p^* \in \mathcal{F} = F(R) \cap F(T_1) \cap \cdots \cap F(T_p)$. Since $\lim_{n \to \infty} d(x_n, \mathcal{F}) = 0$, it follows, as in the proof of Theorem 3.4, that $\{x_n\}$ converges strongly to a common fixed point of the mappings $R, T_1, T_2, \ldots, T_{p-1}, T_p$. This completes the proof.

Corollary 3.7 ([15, Theorem 3.7]). Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$, and $\{x_n\}$ be defined as in (7). Let $R, S, T, U : K \to K$ be four asymptotically nonexpansive mappings satisfying certain condition (GA). If $\mathcal{F} = F(R) \cap F(S) \cap F(T) \cap F(U) \neq \emptyset, 0 < t_1 \leq \beta_n, \beta_n' \leq t_2 < 1$ for some $t_1, t_2 \in (0, 1)$. Suppose one of the mappings in $\{R, S, T, U\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed of the mappings $R, S, T$ and $U$.

Example 3.8 ([9, Example 3.1]). Let $E$ be the real line with the usual norm and let $K = [-1, 1]$. Define $R, T_i : K \to K$ by

$$R(x) = \begin{cases} 
  x & \text{if } x \in [0, 1], \\
  -x & \text{if } x \in [-1, 0) 
\end{cases}, \quad T_1(x) = \begin{cases} 
  -\sin x & \text{if } x \in [0, 1], \\
  \sin x & \text{if } x \in [-1, 0) 
\end{cases}$$
\[
T_i(x) = \begin{cases} 
\frac{x}{i} & \text{if } x \in [0, 1], \\
-\frac{x}{i} & \text{if } x \in [-1, 0) 
\end{cases}; \quad i = 2, 3, \ldots, p
\]

for \( x \in K \). We note that, \( F(R) \cap F(T_1) \cap F(T_2) \cap \cdots \cap F(T_p) = \{0\} \) and \( R, T_1, T_2, \ldots, T_{p-1}, T_p \) are asymptotically nonexpansive. In order to show that \( R \) and \( T_1 \) satisfy (13), we have to consider the following cases:

Case 1. Suppose that \( x \) and \( y \in [0, 1] \). It follows that

\[
|x - T_1 y| = |x + \sin y| = |Rx - T_1 y|;
\]

Case 2. Suppose that \( x \) and \( y \in [-1, 0) \). It follows that

\[
|x - T_1 y| = |x - \sin y| \leq |-x - \sin y| = |Rx - T_1 y|;
\]

Case 3. Suppose that \( x \in [-1, 0) \) and \( y \in [0, 1] \). It follows that

\[
|x - T_1 y| = |x + \sin y| \leq |-x + \sin y| = |Rx - T_1 y|;
\]

Case 4. Suppose that \( x \in [0, 1] \) and \( y \in [-1, 0) \). It follows that

\[
|x - T_1 y| = |x - \sin y| = |Rx - T_1 y|;
\]

Hence (13) is satisfied. Thus Theorems 3.1 and 13 implies that the \( \{x_n\} \) defined by (8) converges both strongly and weakly to 0, respectively.

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**References**


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