Adomian Decomposition Method and its Modification for Nonlinear Abel's Integral Equation

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Abstract

In this paper, by considering the two-step Laplace decomposition method and the appearance of noise terms, exact solutions are calculated for nonlinear Abel's integral equation and generalized Abel's integral equation. This new method provides us to find the solutions with less computation as compared with other methods. The method will be described along with several examples.

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1. Introduction

Weakly singular integral equations model many physical problems. In 1823 Abel derived the equation of motion of the sliding particle along a smooth curve
by singular integral equation. Abel's integral equation also model many physical and biological problems, like radio astronomy, electron emission, radar ranging,

and plasma diagnostics [8-15-12-11]. Abel's integral equations often appears in two nonlinear forms; the first and the second kind are as follows respectively

\[ u(x) = f(x) + \int_0^x \frac{F(u(t))}{\sqrt{x-t}} \, dt \]  

(2)

Where \( f(x) \) is a continuous function, \( 0 \leq x, t \leq T \) and \( T \) is constant. The generalized nonlinear Abel integral equation of the second kind is given by

\[ u(x) = f(x) + \lambda \int_0^x \frac{F(u(t))}{(x-t)^\alpha} \, dt \]  

(3)

where \( \lambda \) is a parameter and \( 0 < \alpha < 1 \). For the linear case, it is assumed that \( F(u(t)) = u(t) \)

In general, nonlinear singular integral equations are very difficult to be solved analytically, but analytical solutions in some special cases can be found in references [17-10-9]. Recently, many powerful methods have been proposed and applied successfully to approximate various types of singular integral equations [1-7-18]. Numerical solutions for the Cauchy and Abel type of weakly singular integral equations are discussed in references [6-2-13]. The recently developed decomposition method proposed by G. Adomian (1923-1996), have been receiving much attention in recent years. Adomian gives the solution as an infinite series usually converging to an exact solution [3-4]. Since this method was first presented in the 1980’s, Adomian's decomposition method has led to several modifications made by various researchers in attempt to improve the accuracy or expand the application of the original method. Moreover, in the study of non-homogenous equations, the noise term phenomenon is rather useful because of the role it plays in the rapid convergence of solutions obtained by ADM. The aim of this paper is to propose ADM and some of its modification to solve Abel's integral equation. The two-step Laplace decomposition method is applied to solve Abel integral equation and generalized Abel's integral equations by considering the noise terms phenomenon.

2. Adomian Decomposition Method for Abel integral equations

In Adomian decomposition method, we consider the functional equation of Abel integral equation of the form
$u = f + Nu$

(4)

where $N$ is a nonlinear operator and $f$ is a given function. Adomian assume the solution as infinite series for the unknown function $u(x)$, given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

(5)

and $Nu$ as the sum of the decomposition series

$$Nu = \sum_{n=0}^{\infty} A_n$$

(6)

where the $A_n$’s are polynomials depending on $u_0, u_1, ..., u_n$ and are called the Adomian polynomials[19], these are defined as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\sum \lambda^i u_i)]_{\lambda=0}, n \geq 0$$

(7)

Substituting (5) and (6) into Abel’s integral equation of the form (4), we get:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \sum_{n=0}^{\infty} A_n$$

The components $u_0(x), u_1(x), ...$ are usually determined by using the recurrence relation

$$u_0(x) = f(x)$$

$$u_{n+1}(x) = A_n(u_0, u_1, ..., u_n), n \geq 0$$

(8)

Having determined the components $u_0, u_1, u_2, ...$ the solution $u(x)$ of equation (4) is determined in the form of a rapid convergent power series by substituting the derived components in equation (5).

**The Noise Terms Phenomenon**

The noise terms that may appear between various components of $u(x)$ are defined as the identical terms with opposite signs [23]. By canceling these noise terms between the component $u_0(x)$ and $u_i(x)$ may give the exact solution that should be justified through substitution. The noise terms, if appeared between components of $u(x)$, accelerate the convergence of the solution and thus minimize the size of the calculations.

**3. Some Recent Modifications of ADM**

Several authors have proposed a variety of modifications to ADM. The modifications arise from evaluating difficulties specific for the type of problem under consideration. In [21] Wazwaz proposed a modification in ADM by
dividing the function \( f(x) \) as the sum of two functions \( f_1(x) \) and \( f_2(x) \) and suggested the following recursive scheme.

\[
\begin{align*}
    u_0(x) &= f_1(x) \\
    u_1(x) &= f_2(x) + A_0 \\
    u_{n+2}(x) &= A_{n+1}(u_0, u_1, \ldots, u_n), n \geq 0
\end{align*}
\]

This modification provides more flexibility to the ADM in solving complicated integral equations [16].

In the new modification, Wazwaz replaces the process of dividing \( f \) into two components. He suggests that \( f \) be expressed in Taylor series

\[
f = \sum_{n=0}^{\infty} f_n
\]

And the new recursive relationship expressed in the form

\[
\begin{align*}
    u_0 &= f_0 \\
    u_{n+1} &= f_{n+1} + A_n, n \geq 0
\end{align*}
\]

It is easily observed that algorithm (10) reduces the number of terms involved in each component, and hence the size of calculations is minimized compared to the standard ADM [22]. Vahidi and Isfahani [20] applied the ADM and the Homotopy Perturbation Method (HPM) to the linear Abel integral equation of the second kind. They showed that the ADM and the HPM for solving the Abel integral equation of the second kind are equivalent. In [16] Kumar et. al. proposed HPM to solve the generalized Abel's integral equation and they showed that ADM and Modified ADM is HPM and Modified HPM with the constructed convex homotopy to solve generalized Abel's integral equation respectively. Another modification introduced by Wazwaz and Mehanna in [24], they combined the Laplace transform method with the ADM for analytic treatment of the nonlinear singular integral equation that described heat transfer.

However, phenomena were recently established to facilitate the convergence of the solution, Adomian and Rach [5] introduced the phenomena of the so-called noise terms. In [14] M. Khan and M. Gondal proposed new modification in standard Laplace decomposition Method and considering the noise terms; their method is called Two-Step Laplace decomposition Method (TSLDA). They applied the method to solve linear Abel's second kind integral equations.

Separating the standard ADM into two steps derives the two-step ADM. The main ideas of two-step ADM are:

set \( f(x) = f_0 + f_1 + \ldots + f_m \)

Based on this, we define
\[ u_0 = f_k + ... + f_{k+s} \]

where \( k = 0,1,...,m \), \( s = 0,1,...,m-k \).

Then we verify that \( u_0 \) satisfies the original equation by substitution. Once the exact solution is obtained we finish; otherwise, we go to the step two.

We set \( u_0 = f \) and continue with the standard Adomian recursive relation (8). Compared to the standard ADM and the modified method, it is seen that the two-step ADM may provide the solution by using one iteration only. Further, the (TSADM) avoids the difficulties arising in the modified method. Furthermore, the number of the terms in \( f \) namely \( m \), is small in many practical problems.

4. A new modification of the Laplace ADM for nonlinear Abel's integral equation

The aim of the present paper is to propose the two-step Laplace ADM to solve nonlinear Abel integral equation. A new Modification of the Laplace ADM consist of the following steps:

First applying the Laplace transform to both sides of equation (2), for example, gives

\[ U (s) = L\{f (x)\} + L\{k (x)\} L\{F (u (x))\} \]  (11)

Where \( L \) is Laplace transform. The ADM defines the solutions \( u(x) \) by the infinite series (5) and similarly

\[ U (s) = \sum_{n=0}^{\infty} U_n (s) \]  (12)

The nonlinear terms \( F(u(x)) \) are usually represented by Adomian polynomials

\[ F (u (x)) = \sum_{n=0}^{\infty} A_n (x) \]  (13)

Substituting (5) and (13) into (11) leads to

\[ L\{\sum_{n=0}^{\infty} u_n (x)\} = L\{f (x)\} + L\{k (x)\} L\{\sum_{n=0}^{\infty} A_n\} \]  (14)

The new modification decomposition method introduces the recursive relation

Applying the inverse Laplace transform to the first part of (15) gives \( u_0 (x) \) that will define \( A_0 \). Using \( A_0 \) will enable the evaluation of \( u_1 (x) \), the determination of \( A_1 \) that will allows the determination of \( u_2 (x) \), and so on. Using (5) the series
solution follows immediately.

Now we illustrate TSLDA after applying the inverse operator. We have \( f(x) \) which can be denoted by another function \( \Phi \) follows

\[
\Phi = f(x)
\]  

(16)

By using TSLDA we set

\[
\Phi = f_0(x) + f_1(x) + \ldots + f_n(x)
\]  

(17)

We define

\[
u_0 = f_k(x) + \ldots + f_{n+k}(x)
\]

(18)

where \( k=0,1,\ldots,m \text{, } s =0,1,\ldots,m-k \). Then we verify that \( u_0 \) satisfies the original equation (2) and by substituting, once the exact solution is obtained we finish. Otherwise, we set \( u_0 = f(x) \) and continue with the standard (LADM)

After applying inverse transform, the exact solution of the equation is found as:

\[
u_{n+1}(x) = L^{-1}\{L\{k(x)\}L\{A_n\}\}, \quad n \geq 0
\]  

(19)

5. Numerical Examples

The following examples are solved to illustrate the efficiency and simplicity of the proposed method.

5.1 Linear Case:

Example (1):

Consider second kind linear Volterra equation in terms of Abel's integral equation

\[
u(x) = x - \frac{4}{3} x^{3/2} + \int_0^x \frac{1}{\sqrt{x-t}} u(t) dt
\]

\[
f(x) = f_0(x) + f_1(x)
\]

Where

\[
f_0(x) = x, \quad f_1(x) = \frac{-4}{3} x^{3/2}
\]
Choosing $u_0 = f_0$ we have

$$u_0(x) = x$$

$$u_1(x) = \frac{-4}{3} x^{3/2} - L^{-1}\left[\frac{\Gamma(\frac{1}{2})}{\sqrt{s}} L[u_0(x)]\right] = 0$$

$$u_{n+1}(x) = 0, n \geq 1$$

Therefore, solution by (TSLDA) is

$$u = \sum_{n=0}^{\infty} u_n(x) = x$$

**Example (2):**
Consider the linear generalized Abel's integral equation

$$u(x) = x^2 + \frac{27}{40} x^{5/3} - \int_{0}^{x} \frac{u(t)}{(u-t)^{5/3}} dt$$

$$u_0 = x^2$$

$$u_1 = \frac{27}{40} x^{5/3} - L^{-1}\left[\frac{\Gamma(\frac{2}{3})}{\sqrt{s}} L[u_0(x)]\right] = 0$$

$$u_{n+1}(x) = 0, n \geq 1$$

Therefore, solution by (TSLDA) is

$$u = \sum_{n=0}^{\infty} u_n(x) = x^2$$

**5.2 The nonlinear Case**

**Example (3):**
Consider the nonlinear Abel integral equation
\[ u(x) = x^{1/2} + \frac{3}{8} x^2 - \int_0^x (x-t)^{-1/2} u^3(t) \, dt \]

\[ u_0 = x^{1/2} \]

\[ u_1 = \frac{3}{8} x^2 - L^{\frac{1}{3}} \left[ \frac{1}{s^{1/2}} L[A_0(x)] \right] = 0 \]

\[ u_{n+1}(x) = 0, \quad n \geq 1 \]

So, the solution is

\[ u(x) = x^{1/2} \]

Note that \( A_0 \) is Adomian polynomial of the nonlinear term \( u^3 \) which compute from relation (7)

**Example (4):**

Consider the generalized nonlinear Abel integral equation:

\[ u(x) = (x + x^2)^{4} - \frac{9}{40} x^{5/3} (3x + 4) + \int_0^x \frac{u^{1/2}(t)}{(x-t)^{1/3}} \, dt \]

\[ u_0 = (x + x^2)^{4} \]

\[ u_1 = -\frac{9}{40} x^{5/3} (3x + 4) + L^{\frac{3}{25}} \left[ \frac{1}{s^{1/2}} L[A_0] \right] = -0.675 x^{9/5} - 0.9 x^{5/3} + 0.675 x^{9/5} \]

\[ + 1.104545 x^{11/3} + 0.47337 x^{14/3} \]

In canceling the noise terms from \( u_1(x) \), \( u_1(x) \) contain more terms. Then it is necessary to compute more components of \( u(x) \) to determine the solution. However not all non-homogenous equations have the noise terms phenomenon.

**6. Conclusion**

In this work, we proposed new modification in Laplace decomposition method we have established solutions for nonlinear Abel's integral equation and generalized Abel's integral equation by this method. Our method introduces an efficient algorithm that improves the performance of the standard Laplace ADM for solving nonlinear Abel integral equation. The method overcomes the difficulties arising in calculating the Adomian polynomials. The efficiency of the method was tested on some numerical examples, and the results show that the method is easier than many other numerical techniques.
Adomian decomposition method

References


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