Solutions to Semipositone Eigenvalue Problems

Qinfu Sun

School of Mathematical Sciences
Qufu Normal University, Qufu, 273165, China
sqf@mail.qfnu.edu.cn

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Abstract

In this paper, we investigate a class of third-order three-point semipositone eigenvalue problems under the conditions that the nonlinear term is continuous, semipositone and lower unbounded.

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1 Introduction

In this paper, we study the following third-order three-point semipositone eigenvalue boundary value problems (SEBVP):

\[
\begin{align*}
&u'''(t) - \lambda f(t, u) = 0, \quad t \in (0, 1); \\
&u(0) = u'(\eta) = u''(1) = 0,
\end{align*}
\]

where \(\lambda > 0\) is a positive parameter, \(1/2 < \eta < 1\), \(f(t, u) : (0, 1) \times [0, +\infty) \to (-\infty, +\infty)\).

In recent years, the existence of positive solutions for nonlinear boundary value problems received wide attention. But they all request the positive continuous or lower bounded of the nonlinear term (see [1-8]). The motivation for the present paper does back to a pioneering paper by Anuradha [1], which has been so influential as to motivate several authors to develop further theory of [1] in other directions. For example, in 1998, D. Aanderson [3] considered the
problem (1.1) and obtained an existence result about positive solutions when
\( f(t, l) = g(l) \) and \( g : [0, +\infty) \to [0, +\infty) \). Recently, Yao [4] has investigated
(1.1) when \( f \) is semipositone and lower bounded, and he obtained the following
existence theorem.

**Theorem A.** Suppose that
1. \( \inf \{ f(t, l) : (t, l) \in [0, 1] \times [0, +\infty) \} = -M > -\infty \), where \( M \geq 0 \).
2. \( B = \max \{ f(t, l) : (t, l) \in [0, 1] \times [0, 1] \} + M \geq 0 \).
3. There exist \( 0 < \alpha < \beta < 1 \) such that \( \lim_{t \to +\infty} \min_{\alpha \leq t \leq \beta} f(t, l) = +\infty \).

Then the problem (1.1) has at least one positive solution, provided
\( 0 < \lambda < \min \left\{ \frac{6}{B\eta^2(3 - 2\eta)}, \frac{6(2\eta - 1)}{M[1 - 3(1 - \eta)^2]}, \frac{1}{M} \right\} \).

For semipositone nonlinear problem, Kosmatov [5] make the following as-\nsumptions for nonlinear term \( f \):
- \( (A_1) \ f(t, z) \) is a continuous function on \( [0, 1] \times [0, \infty) \);
- \( (A_2) \) There exists \( M > 0 \) such that \( f(t, z) + M \geq 0 \) on \( [0, 1] \times [0, \infty) \);
- \( (A_3) \) There exist continuous nonnegative nondecreasing on \( [0, \infty) \) functions\n\( \psi_a(z) \) and \( \psi_b(z) \) with \( \psi_b(z) \leq f(t, z) + M \leq \psi_a(z) \) on \( [0, 1] \times [0, \infty) \).

All the above mentioned paper, the authors discuss the semipositone eigen-
value boundary value problem (SEBVP) under the key conditions that there
exists \( M > 0 \) such that \( f \geq -M \) or/and there exists upper control-function
for nonlinear term.

In this paper, we delete the restriction on lower bounded and on upper
control-function of the nonlinear term, \( f(t, u) : [0, 1] \times (0, +\infty) \to (-\infty, +\infty) \) is continuous, i.e., we allow that the nonlinear term \( f \) is both semipositone
and lower unbounded is guaranteed.

Our main tool of this paper is the following fixed point index theory.

**Theorem 1.1**[6,7]. Suppose \( E \) is a real Banach space, \( K \subset E \) is a cone,\nlet \( \Omega_r = \{ u \in K : \| u \| \leq r \} \). Let operator \( T : \Omega_r \to K \) be completely
continuous and satisfy \( Tx \neq x, \forall x \in \partial \Omega_r \). Then:
- (i) If \( \| Tx \| \leq \| x \|, \forall x \in \partial \Omega_r \), then \( i(T, \Omega_r, K) = 1 \);
- (ii) If \( \| Tx \| \geq \| x \|, \forall x \in \partial \Omega_r \), then \( i(T, \Omega_r, K) = 0 \).

## 2 Preliminary Notes

Let \( I = [0, 1] \), \( E = C[I, R] \), then \( E \) is a Banach space with norm \( \| x \| = \max_{t \in I} |x(t)| \). We also introduce the space \( L^1(0, 1) \) with norm \( \| x \|_1 = \int_0^1 |x(t)| dt \).
Throughout this paper, we shall use the following notation:

\[
G(t, s) = \begin{cases} 
    ts - \frac{1}{2}t^2, & 0 \leq s \leq \eta, 0 \leq t \leq s; \\
    \frac{1}{2}s^2, & 0 \leq s \leq \eta, 0 \leq s \leq t; \\
    \eta t - \frac{1}{2}t^2, & \eta \leq s \leq 1, 0 \leq t \leq s; \\
    \frac{1}{2}s^2 - ts + \eta t, & \eta \leq s \leq 1, 0 \leq s \leq t.
\end{cases}
\]

It is well known that \(G(t, s)\) is the Green’s function of homogeneous boundary value problem:

\[
\begin{align*}
u'''(t) &= 0, \quad 0 \leq t \leq 1; \\
u(0) &= u'(\eta) = u''(1) = 0.
\end{align*}
\]

Obviously, \(G(t, s)\) is nonnegative continuous function.

By direct account, we can easily obtain the following results.

**Lemma 2.1** ([4]). \(G(t, s)\) defined as above have the following properties:

\[
q(t)J(s) \leq G(t, s) \leq J(s), \quad 0 \leq t, s \leq 1,
\]

where

\[
J(s) = \max_{t \in I} G(t, s) = \begin{cases} 
    \frac{1}{2}s^2, & 0 \leq s \leq \eta, \\
    \frac{1}{2}\eta^2, & \eta \leq s \leq 1,
\end{cases} \quad q(t) = \begin{cases} 
    \eta t, & 0 \leq t \leq \eta; \\
    2\eta t - t^2, & \eta \leq t \leq 1,
\end{cases}
\]

**Lemma 2.2.** For the unique position solution \(u(t)\) of the following BVP:

\[
\begin{align*}
u'''(t) &= h(t), \quad 0 < t < 1; \\
u(0) &= u'(\eta) = u''(1) = 0,
\end{align*}
\]

where \(h \in L^1(0, 1), \ h \geq 0\). Then \(u(t) \geq \|u\|q(t), \ 0 \leq t \leq 1\).

**Proof.** By \(q(t)J(s) \leq G(t, s) \leq J(s), \ 0 \leq t, s \leq 1\), we have

\[
u(t) = \int_0^1 G(t, s)h(s)ds \leq \int_0^1 J(s)h(s)ds,
\]

so, \(\|u\| \leq \int_0^1 J(s)h(s)ds\). Therefore, for \(0 \leq t \leq 1\), we have

\[
u(t) = \int_0^1 G(t, s)h(s)ds \geq q(t) \int_0^1 J(s)h(s)ds \geq \|u\|q(t).
\]

This completes the proof of Lemma 2.2.

**Lemma 2.3.** For the unique position solution \(u(t)\) of the following BVP:

\[
\begin{align*}
u'''(t) &= h(t), \quad 0 < t < 1; \\
u(0) &= u'(\eta) = u''(1) = 0,
\end{align*}
\]
where \( h \in L^1(0, 1), h \geq 0 \). Then, for any \( \theta \in (0, 1/2) \), there exists constant \( \delta > 0 \) such that \( u(t) \geq \delta \| u \|, \ \theta \leq t \leq 1 - \theta \).

**Proof.** Let \( \delta = \max_{\theta \leq t \leq 1-\theta} q(t) \), then by the Lemma 2.2, we can obtain the results. This completes the proof of Lemma 2.3.

**Lemma 2.4.** Suppose that \( \overline{w}(t) \) is the solution of the following BVP,

\[
\begin{align*}
&\begin{cases}
    u''(t) = M(t), \quad t \in (0, 1); \\
    u(0) = u'(\eta) = u''(1) = 0,
\end{cases}
\end{align*}
\]

where \( M(t) \in L^1(0, 1), \ M(t) > 0 \). Then, there constant \( C \geq 1 \) such that

\[
\overline{w}(t) \leq C\| M \|_1 q(t), \ 0 \leq t \leq 1.
\]

**Proof.** For \( t \in [\eta, 1] \), we can have

\[
\overline{w}(t) = \int_0^1 G(t, s)M(s)ds
\]

\[
= \int_0^\eta \frac{1}{2}s^2 M(s)ds + \int_\eta^t (\frac{1}{2}s^2 - ts + \eta t) M(s)ds + \int_t^1 (\eta t - \frac{1}{2}t^2) M(s)ds
\]

\[
\leq \int_0^\eta \frac{1}{2}s^2 M(s)ds + \int_\eta^t (\frac{1}{2}s^2 - ts + \eta t) M(s)ds + \int_t^1 (\eta t - \frac{1}{2}t^2) M(s)ds
\]

\[
\leq \left[ \frac{1}{2}t^2 + \left( \frac{1}{2}t^2 - t\eta + \eta t \right) + 2(\eta t - \frac{1}{2}t^2) \right] \int_0^1 M(s)ds
\]

\[
\leq 2\eta t \int_0^1 M(s)ds \leq 3(2\eta t - t^2)\| M \|_1.
\]

In fact, by \( 1/2 < \eta \leq t \leq 1 \), we have \( 3(2\eta t - t^2) - 2\eta t = 4\eta t - t^2 \geq 4\eta^2 - t^2 \geq 0 \).

For \( t \in [0, \eta] \), we can have

\[
\overline{w}(t) = \int_0^1 G(t, s)M(s)ds
\]

\[
= \int_0^t \frac{1}{2}s^2 M(s)ds + \int_t^\eta (ts - \frac{1}{2}t^2) M(s)ds + \int_\eta^1 (\eta t - \frac{1}{2}t^2) M(s)ds
\]

\[
\leq \int_0^t \frac{1}{2}s^2 M(s)ds + \int_t^\eta (ts - \frac{1}{2}t^2) M(s)ds + \int_\eta^1 (\eta t - \frac{1}{2}t^2) M(s)ds
\]

\[
\leq \left[ \frac{1}{2}t^2 + \eta t + (\eta t - \frac{1}{2}t^2) \right] \int_0^1 M(s)ds
\]

\[
= 2\eta t \int_0^1 M(s)ds \leq 2\| M \|_1 \eta t.
\]

Then, we choose constant \( C = 3 > 1 \), by the above, we have

\[
\overline{w}(t) \leq C\| M \|_1 q(t), \ 0 \leq t \leq 1.
\]
This completes the proof of Lemma 2.4.

In the rest of the paper, we also make the following assumptions:

(H) \( f \in C([0, 1] \times [0, +\infty), [-\infty, +\infty)) \), and there exists function \( M(t) \in L^1(0, 1) \), \( M(t) > 0 \) and \( 0 < \int_0^1 J(s)M(s)ds < \infty \) such that \( f(t, u) \geq -M(t), \forall t \in (0, 1), u \geq 0 \), where \( J(s) \) is defined in Lemma 2.1.

By Lemma 2.3, for \( \theta \in (0, 1/2) \), we denote a cone \( K \) of \( E \):

\[
K = \{ u \in E : u(t) \geq \|u\|_q(t), \theta \leq t \leq 1 - \theta \},
\]

For convenience, we set

\[
k_0 = \min_{\theta < t, s < 1-\theta} G(t, s), \quad K_0 = \max_{0 < t, s < 1} G(t, s), \quad \delta = \min_{\theta \leq t \leq 1-\theta} q(t),
\]

\[
f_0 = \lim_{u \to 0} \max_{0 \leq t \leq 1} \frac{f(t, u)}{u}, \quad f_\infty = \lim_{u \to \infty} \min_{0 \leq t \leq 1} \frac{f(t, u)}{u}.
\]

## 3 Main Results

In this section, we present our main results.

**Theorem 3.1.** Suppose that condition (H) hold. Then, if \( 0 < 2K_0 f_0^1 (f_0 + M(s))ds < k_0 \delta f_\infty < \infty \), then for each \( \lambda \in \left( 0, \frac{2K_0 f_0^1 (f_0 + M(s))ds}{k_0 \delta f_\infty} \right) \), the SEBVP (1.1) has at least one positive solution.

**Corollary 3.2.** Suppose that condition (H) hold. Then, If \( f_0 = 0 \) and \( f_\infty = \infty \), then for any \( \lambda \in \left( 0, \frac{1}{K_0 \|M\|_1} \right) \), the SEBVP (1.1) has at least one positive solution.

**Corollary 3.3.** Suppose that condition (H) hold. Then, If \( f_\infty = \infty \), \( 0 < f_0 < \infty \), then for each \( \lambda \in \left( 0, \frac{1}{K_0 \int_0^1 (f_0 + M(s))ds} \right) \), the SEBVP (1.1) has at least one positive solution.

**Corollary 3.4.** Suppose that condition (H) hold. Then, If \( f_0 = 0 \), \( 0 < f_\infty < \infty \), then for each \( \lambda \in \left( 0, \frac{1}{k_0 \delta f_\infty} \right), \frac{1}{K_0 \|M\|_1} \), the SEBVP (1.1) has at least one positive solution.

**The proof of Theorem 3.1.** By Lemma 2.4, we set \( w(t) = \overline{w}(t) \). Then \( u(t) \) is the positive solutions of the SEBVP (1.1) if and only if \( \overline{u}(t) = u(t) + w(t) \) is the positive solutions of the EBVP

\[
\begin{align*}
\begin{cases}
u''(t) - \lambda F(t, u(t) - w(t)) = 0, & t \in (0, 1); \\
u(0) = u'(\eta) = u''(1) = 0,
\end{cases}
\end{align*}
\]

and \( \overline{u}(t) \geq w(t), t \in I \), where for \( t \in I \),

\[
F(t, u) = H(t, u) + M(t), \quad H(t, u) = \begin{cases} f(t, u), & u \geq 0, \\
f(t, 0), & u < 0,
\end{cases}
\]
Obviously, EBVP (3.1) is equivalent to the equation

\[ u(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds. \quad (3.2) \]

and consequently, it’s solution is equivalent to the fixed point problem \( u = Tu \) with operator \( T : E \to E \) given by

\[ (Tu)(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds. \quad (3.3) \]

Then we shall divide the rather long proof into three steps.

(I) \( T : K \to K \) is completely continuous.

(a) Firstly, we proof that \( T(K) \subset K \). By Lemma 2.1, (3.4), for any \( u(t) \in K, \ t \in J \), we have

\[ (Tu)(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds \leq \int_0^1 J(s)\lambda F(s, u(s) - w(s))ds. \]

So,

\[ \|Tu\| \leq \int_0^1 J(s)\lambda F(s, u(s) - w(s))ds. \quad (3.4) \]

Then, by Lemma 2.1 and (3.4), for \( \theta \leq t \leq 1 - \theta, \ u \in K \), we have

\[ (Tu)(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds \geq q(t) \int_0^1 J(s)\lambda F(s, u(s) - w(s))ds \geq \|Tu\|q(t). \]

Then \( T(K) \subset K \).

(b) Secondly, we will show that \( T \) ia compact operator. Let \( D \subset K \) be any bounded set, then there exists a constant \( M > 0 \) such that \( \|u\| \leq M, \ u \in D \). Then, we have

\[ \|(Tu)(t)\| \leq \int_0^1 J(s)\lambda (L + M(s))ds. \]

where \( L = \sup_{0 \leq t \leq 1, \|u\| \leq M} H(t, u) \). Therefore, \( T(D) \) is uniformly bounded.

Next, we will show \( |(Tu)'| \in L^1[0, 1], \ u \in D \). In fact, by (3.3), we know that if \( t \in [\eta, 1] \), we can get

\[ |(Tu)'(t)| = \left| \int_\eta^t (\eta - s)\lambda F(s, u(s) - w(s))ds + \int_t^1 (\eta - t)\lambda F(s, u(s) - w(s))ds \right| \]

\[ \leq \left| \int_\eta^t (\eta - s)\lambda (L + M(s))ds + \int_t^1 (\eta - t)\lambda (L + M(s))ds \right| \]

\[ \leq (L + 1)\lambda \left( \int_\eta^t (s - \eta)(1 + M(s))ds + \int_t^1 (t - \eta)(1 + M(s))ds \right) \]

\[ =: (L + 1)\lambda h(t), \]

where \( h(t) = \int_\eta^t (s - \eta)(1 + M(s))ds + \int_t^1 (t - \eta)(1 + M(s))ds \).
Then, we have
\[
\int_0^1 |h(t)| dt = \int_0^1 \left| \int_0^t (s-\eta)(1+M(s))ds + \int_0^t (t-\eta)(1+M(s))ds \right| dt
\]
\[
= \int_0^1 (s-\eta)(1+M(s))ds \int_0^1 dt + \int_0^1 (1+M(s))ds \int_0^1 (t-\eta)dt
\]
\[
\leq \int_0^1 (s-\eta)(1+M(s))ds \int_0^1 dt + \int_0^1 (1+M(s))(\frac{1}{2}s^2 - \eta s)ds
\]
\[
\leq \int_0^1 (1+M(s))ds < \infty.
\]

Then, \(0 \leq \int_0^1 |(Tu)'(t)| dt < \infty\).

Similar to the above, for \(t \in [0, \eta]\), we can also get \(0 \leq \int_0^1 |(Tu)'(t)| dt < \infty\).

Then, for any \(0 \leq t_1 \leq t_2 \leq 1, u \in D\), we have
\[
|(Tu)(t_1) - (Tu)(t_2)| = \left| \int_{t_1}^{t_2} (Tu)'(t) dt \right| \leq \int_{t_1}^{t_2} |(Tu)'(t)| dt.
\]

So by the absolute continuity of the integral, we know that \(T(D)\) is equicontinuous on \([0,1]\). Thus, according to Ascoli-Arzela’s theorem, we know that \(T(D)\) is a relatively compact set, i.e., \(T\) is compact operator.

In the following, we show that \(T\) is continuous. Assume \(y_n, y_0 \in K\) such that \(\|y_n - y_0\| \to 0\), \((n \to +\infty)\). Then there exists \(L_2 > 0\) such that \(\|y_n\| \leq L_2\) and \(\|y_0\| \leq L_2\).

And further by Lemma 2.1, for any \(t \in [0,1]\),
\[
|Ty_n(t) - Ty_0(t)|
\]
\[
= \left| \lambda \int_0^1 G(t,s)F(s,[y_n(s) - w(s)]) - F(s,[y_0(s) - w(s)]) ds \right|
\]
\[
\leq \lambda \int_0^1 J(s) \left| f(s,[y_n(s) - w(s)]) - f(s,[y_0(s) - w(s)]) \right| ds.
\]

So, we set \(Y_n(s) := J(s)[f(s,[y_n(s) - x(s)]) - f(s,[y_0(s) - x(s)])]\).

Next, we will show that \(Y_n(s) \to 0\), \((n \to +\infty)\) for any fixed \(s \in (0,1)\).

In fact, in view of the continuity of \(f\) with respect to \(u\), for any \(\varepsilon > 0\), there exists a constant \(\delta > 0\) such that for any \(v_1, v_2 \geq 0\), if \(|v_1 - v_2| < \delta\), we have
\[
\left| f(s,v_1) - f(s,v_2) \right| < \frac{\varepsilon}{J(s)}.
\]

By \(y_n(s) \to y_0(s)\), there exists a constant \(N > 0\) such that \(|y_n(s) - y_0(s)| < \delta\) for \(n > N\). Noting that
\[
\left| [y_n(s) - w(s)] - [y_0(s) - w(s)] \right| = |y_n(s) - y_0(s)|,
\]
so for \( n > N \), we have that
\[
\left| f(s, [y_n(s) - w(s)]) - f(s, [y_0(s) - w(s)]) \right| < \frac{\varepsilon}{J(s)}.
\]

Thus, for any fixed \( s \in (0, 1) \), and for any \( \varepsilon > 0 \), \( \exists N > 0 \) such that as \( n > N \), we have
\[
\left| Y_n(s) - 0 \right| = J(s)\left| f(s, [y_n(s) - w(s)]) - f(s, [y_0(s) - w(s)]) \right| < \varepsilon,
\]
i.e., for any fixed \( s \in (0, 1) \), we have \( Y_n(s) \to 0 \), \( (n \to \infty) \).

Furthermore, by using Lebesgue dominated convergence theorem and the above mentioned, we can easily obtain that
\[
\|Ty_n - Ty_0\| \leq \int_0^1 Y_n(s)ds \to 0, \ (n \to +\infty).
\]
Therefore, \( T : K \to K \) is continuous, and thus is a completely continuous operator.

(II) Next, we will discuss the positive solution of the EBVP (3.1).

By the definition of \( f_0 \) and \( \frac{2}{\kappa_0f_\infty} < \lambda < \frac{1}{\kappa_0\int_0^1(f_0 + M(s))ds} \), we have that there exist \( r \geq 1 \) and \( \varepsilon > 0 \) such that: \( f(t, u) \leq (f_0 + \varepsilon)u \), \( 0 \leq u \leq r \), \( t \in [0, 1] \),
\[
\frac{2}{\kappa_0\delta(f_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{\kappa_0\int_0^1(f_0 + \varepsilon + M(s))ds}.
\]
Set \( \Omega_1 = \{ u \in E : \| u \| \leq r \} \). Then, for \( t \in [0, 1] \) and \( u \in \partial\Omega_1 \), we have
\[
u(t) - w(t) = u(t) - \overline{w}(t) \geq u(t) - C\|M\|_1q(t) \geq u(t) - \frac{C\|M\|_1}{r}u(t) \geq 0,
\]
and \( u(t) - w(t) \leq \| u \| = r \), i.e., \( \| u(t) - w(t) \| \leq r \). So, for \( t \in [0, 1] \) and \( u \in \partial\Omega_1 \), we have
\[
(Tu)(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds
\]
\[
= \int_0^1 G(t, s)\lambda(\int_0^1 (f(s, u(s) - w(s)) + M(s))ds
\]
\[
\leq K_0\lambda \int_0^1 ((f_0 + \varepsilon)r + M(s))ds
\]
\[
\leq K_0r\lambda \int_0^1 (f_0 + \varepsilon + M(s))ds \leq r = \| u \|.
\]
Therefore, we have \( \| Tu \| \leq \| u \|, \ \forall u \in K \cap \partial\Omega_1 \). Then by Theorem 1.1, we have
\[
i(T, \Omega_1, K) = 1.
\] (3.5)
On the other hand, by the definition of \( f_\infty \) and for the above \( \varepsilon > 0 \), we have that there exist \( l \geq 1 \) such that: \( f(t, u) \geq (f_\infty - \varepsilon)u, \ u \geq l, \ t \in [0, 1] \).

Set \( R > \max \{ r, 2C\|M\|_1, 2l/\delta \} \) and \( \Omega_2 = \{ u \in E : \|u\| \leq R \} \). Then, for \( t \in [0, 1] \) and \( u \in \partial \Omega_2 \), we have

\[
u(t) - w(t) = u(t) - \overline{w}(t) \geq u(t) - C\|M\|_1q(t) \geq u(t) - \frac{C\|M\|_1}{R}u(t) \geq \frac{1}{2}u(t),\]

so, for \( t \in [\theta, 1 - \theta] \), we have \( u(t) - w(t) \geq \frac{1}{2}u(t) \geq \frac{\|u\|}{2}q(t) \geq \frac{\delta R}{2} \geq l \).

So, for \( t \in [0, 1] \), \( u \in \partial \Omega_2 \) and Lemma 2.3, we have

\[
(Tu)(t) = \int_0^1 G(t, s)\lambda F(s, u(s) - w(s))ds
\]
\[
= \int_0^1 G(t, s)\lambda(f(s, u(s) - w(s)) + M(s))ds
\]
\[
\geq k_0\lambda \int_\theta^{1-\theta} ((f_\infty - \varepsilon)(u(s) - w(s)) + M(s))ds
\]
\[
\geq \frac{1}{2}k_0\lambda \int_0^1 (f_\infty - \varepsilon)u(s)ds
\]
\[
\geq \frac{1}{2}k_0\lambda(f_\infty - \varepsilon) \int_0^1 \delta\|u\|ds \geq \|u\|.
\]

Therefore, we have \( \|Tu\| \geq \|u\|, \ \forall \ u \in K \cap \partial \Omega_2 \). Then by Theorem 1.1, we have

\[
i(T, \Omega_2, K) = 0. \quad (3.6)
\]

Therefore, by (3.5), (3.6), \( r < R \), we have \( i(T, \Omega_2 \setminus \overline{\Omega_1}, K) = -1 \). Then operator \( T \) has a fixed point \( \bar{u} \in K \cap (\Omega_2 \setminus \overline{\Omega_1}) \), and \( r \leq \|\bar{u}\| \leq R \).

(III) Finally, we will show that \( \bar{u}(t) \geq w(t), \ \varepsilon \leq \|\bar{u}\| \leq R \).

By, Lemma 2.3 and 2.4, for \( t \in (\theta, 1 - \theta) \), we have

\[
\bar{u}(t) \geq \|\bar{u}\|q(t) \geq rq(t) > C\|M\|_1q(t) \geq \overline{w}(t) = w(t),
\]

i.e., \( u(t) = \bar{u}(t) - w(t) \) is the positive solution of SBVP (1.1). This completes the proof of Theorem 3.1.

The corollary (3.2), (3.3), (3.4) are the direct of the Theorem(3.1).

4 Example

Example 5.1. Consider the following semipositone eigenvalue boundary value problem (SEBVP):

\[
\begin{align*}
u'''' - \lambda \left( u(t) \ln(1 + u(t)) - \frac{1}{2}t^2 \right) &= 0, \quad 0 < t < 1, \\
u(0) = u'(2/3) = u''(1) = 0.
\end{align*}
\]
We can easily show that \( f(t, u) = u(t) \ln(1 + u(t)) - \frac{1}{2} t^2 \) satisfy:

\[
f(t, u) = u(t) \ln(1 + u(t)) - \frac{1}{2} t^2 \geq -\frac{1}{2} t^2 = -M(t),
\]

So, condition (H) holds.

Next, we can easily know that \( f_0 = 0, f_{\infty} = \infty \).

By direct account, we can easily obtain

\[
K_0 = \frac{2}{9}, \|M\|_1 = \int_0^1 M(s)ds = \int_0^1 \frac{1}{2} t^2 dt = \frac{1}{6}.
\]

So, by the case (2) in Theorem 3.1, we can show that for \( \lambda \in (0, \frac{1}{K_0 \|M\|_1}) = (0, 27) \), the SEBVP (5.1) have at least one positive solution.

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References


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