Potential Symmetries and Differential Forms

for Wave Dissipation Equation

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Abstract

The differential form method is applied to study the wave equation with dissipation and extended to determine potential symmetries. The symmetry group are given and group invariant solutions associated to the symmetries are obtained.

Keywords: Differential-form, Symmetry-generators

1. Introduction

The method of writing system of differential equation in terms of differential forms and finding their symmetries was introduced by Harrison and Estabrook [11]. Papachristou and Harrison [17-19], generalized the method to vector valued or Lie algebra – valued differential forms and used in two-dimensional Dirac equation and the Yang – Mills free field equations in Minkowski space-time. Edeled [3-6], developed the theory of differential forms and explore the use of differential form in physics. Gupta and Sharma [8] had worked on nonlinear diffusion equation with convection term. Haager et. al. [9] are analyzes the symmetries of a class of nonlinear telegraph equations are examined. Pakdemirli et. al. [15, 16] considered boundary layer equations for non-Newtonian fluids. Davison and Kara [2] treated Burgers equation to obtain potential and approximate symmetries using differential form method. A generalized nonlinear Shröndinger equation with attention to both symmetries and Bäcklund
transformations were considered by Harnad and Winternitz [10]. Web et. al. [20, 21] are analyzes a nonlinear magnetic potential equation with conservation laws, the Liouville equation and nonlinear Shrödinger equations for a type of MHD waves using differential form method. Kara et. al. [12], was studied the nonlinear wave equation with variable long wave velocity and the Gordon-type equations (in particular, the $\varphi^4$-model equation). Ozer and Suhubi [14] considered nonvacuum Maxwell equations with nonlinear constitutive relations. In the present study, we obtained the determined equations for system of linear differential equation for wave equation with dissipation. And constricted a range of symmetry generators, translation and scaling etc. are related to conservation the laws, with the aid of these symmetries explicit new solution are derived. Finally, the group invariant solutions are obtained for all the cases for system of differential equation.

2. Determined Equations of system of linear differential equation

2.1: Consider the linear wave dissipation equation $u_{tt} + u_t = u_{xx}$ (2.1.1)

For consideration of differential form of equation (2.1.1), we consider the following auxiliary system: $v = u_x, w = u_t, w + w_t = u_x$ (2.1.2)

we introduce the following 2-forms:

$$\alpha = du \ dt - v \ dx \ dt$$

$$\beta = du \ dx - w \ dt \ dx$$

$$\gamma = w \ dx \ dt + dw \ dx + dv \ dt$$

which gives the system (2.1.2) when annulled. Here we drop the wedge product $\wedge$ to save writing. Consider the symmetry of equation (2.1.1) in the form

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial v} + \psi \frac{\partial}{\partial w}$$

Lie derivatives of $\alpha$, $\beta$ and $\gamma$ as

$$L_X \alpha = X \int \alpha$$

$$L_X \beta = X \int \beta$$

$$L_X \gamma = X \int \gamma$$

The Lie derivatives of $\beta$ as

$$L_X \beta = X \int \beta$$

$$L_X \gamma = X \int \gamma$$

(2.1.3)

(2.1.4)
The Lie derivatives of $\gamma$ as

$$L_X \gamma = X \left( \int \gamma d \gamma + d \left( \int \gamma \right) \right)$$

$$L_X \gamma = X \left( \left( dw \ dx \ dt \right) + d \left( w \xi \ dt - w \tau dx + \psi \ dx - \xi \ dw + \phi \ dt - \tau \ dv \right) \right)$$

$$L_X \gamma = \left( - w \xi - w \tau_i \phi_i + \psi_i - \psi \right) dt \ dx + \left( - w \xi_u - \phi_u \right) dt \ du + \left( - w \xi_v - \phi_v \right) dt \ dv + \left( - w \xi_w - \phi_w \right) dw \ dx + (w \tau_i - \psi_i) dx \ du + (w \tau_u - \psi_u) dx \ dv + (w \tau_v - \psi_v - \tau_i) dx \ dw + (w \tau_w - \psi_w - \xi_k) \ dv \ dw - \tau_u \ du - \xi_u \ du \ dw + (\tau_u - \xi_u) \ dv \ dw$$

From equation (2.1.3), (2.1.4) and (2.1.5), we have the following system of determination equations table

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dt \ dx$</td>
<td>$- \theta_x + v \xi_x + v \tau_i + \phi = 0$</td>
</tr>
<tr>
<td>$dt \ du$</td>
<td>$- \theta_u + v \xi_u - \tau_i = 0$</td>
</tr>
<tr>
<td>$dt \ dv$</td>
<td>$- \theta_v + v \xi_v = 0$</td>
</tr>
<tr>
<td>$dt \ dw$</td>
<td>$- \theta_w + v \xi_w = 0$</td>
</tr>
<tr>
<td>$dx \ du$</td>
<td>$- \tau_x - v \tau_u = 0$</td>
</tr>
<tr>
<td>$dx \ dv$</td>
<td>$- v \tau_v = 0$</td>
</tr>
<tr>
<td>$dx \ dw$</td>
<td>$- v \tau_w = 0$</td>
</tr>
<tr>
<td>$du \ dv$</td>
<td>$\tau_v = 0$</td>
</tr>
<tr>
<td>$du \ dw$</td>
<td>$\tau_u = 0$</td>
</tr>
<tr>
<td>$dt \ dx$</td>
<td>$\theta_i - w \xi_i - w \tau_i = 0$</td>
</tr>
<tr>
<td>$dt \ du$</td>
<td>$- w \xi_u = \xi_i = 0$</td>
</tr>
<tr>
<td>$dt \ dv$</td>
<td>$- w \xi_v = 0$</td>
</tr>
<tr>
<td>$dt \ dw$</td>
<td>$- w \xi_w = 0$</td>
</tr>
<tr>
<td>$dx \ du$</td>
<td>$- \theta_u - \xi_x - w \tau_u = 0$</td>
</tr>
<tr>
<td>$dx \ dv$</td>
<td>$- \theta_v + w \tau_v = 0$</td>
</tr>
<tr>
<td>$dx \ dw$</td>
<td>$- \theta_w + w \tau_w = 0$</td>
</tr>
<tr>
<td>$du \ dv$</td>
<td>$\xi_v = 0$</td>
</tr>
<tr>
<td>$du \ dw$</td>
<td>$\xi_w = 0$</td>
</tr>
<tr>
<td>$dt \ dx$</td>
<td>$- w \xi_x - w \tau_i - \phi_i + \psi_i - \psi = 0$</td>
</tr>
<tr>
<td>$dt \ du$</td>
<td>$- w \xi_u - \phi_u = 0$</td>
</tr>
<tr>
<td>$dt \ dv$</td>
<td>$- w \xi_v - \phi_v - \tau_i = 0$</td>
</tr>
<tr>
<td>$dt \ dw$</td>
<td>$- w \xi_w - \phi_w - \xi_i = 0$</td>
</tr>
<tr>
<td>$dx \ du$</td>
<td>$w \tau_u - \psi_u = 0$</td>
</tr>
<tr>
<td>$dx \ dv$</td>
<td>$w \tau_v - \psi_v - \tau_u = 0$</td>
</tr>
<tr>
<td>$dx \ dw$</td>
<td>$w \tau_w - \psi_w - \xi_u = 0$</td>
</tr>
<tr>
<td>$du \ dv$</td>
<td>$- \tau_u = 0$</td>
</tr>
<tr>
<td>$du \ dw$</td>
<td>$- \xi_w = 0$</td>
</tr>
<tr>
<td>$dv \ dw$</td>
<td>$\tau_w - \xi_v = 0$</td>
</tr>
</tbody>
</table>
From the determinate equation we obtain:
\[
\tau_x = 0, \quad \tau_u = 0, \quad \tau_v = 0, \quad \tau_w = 0, \quad \xi_x = 0, \quad \xi_u = 0, \quad \xi_v = 0, \quad \xi_w = 0, \quad \chi_v = 0, \quad \theta_v = 0, \quad \theta_w = 0, \quad \phi_u = 0, \quad \phi_v = 0, \quad \psi_u = 0, \quad \psi_v = 0
\]
and from the equations (2), (14), (21) and (25)
\[
\theta_u = \phi_v = \psi_w = -\xi_x = -\tau_t
\]
(2.1.6)

From equation (2.1.6) we have
\[
\theta_{ut} = \phi_{vt} = \psi_{wt} = -\xi_{xt} = -\tau_u = 0, \quad \theta_{ux} = \phi_{vx} = \psi_{wx} = -\xi_{xx} = -\tau_{tx} = 0, \\
\theta_{uv} = \phi_{vy} = \psi_{wy} = -\xi_{xw} = -\tau_{tw} = 0, \\
\theta_{uw} = \phi_{vy} = \psi_{ww} = -\xi_{ww} = -\tau_{ww} = 0,
\]
(2.1.7)

Adding the equations (2) and (21) - (2\theta_u + \phi_v) - 2\tau_t = 0 which gives
\[
\tau_t = -\frac{1}{2} (\theta_u + \phi_v)
\]
(2.1.8)

from equations (14) and (25) - (\theta_u + \phi_v) - 2\xi_x = 0 which gives
\[
\xi_x = -\frac{1}{2} (\theta_u + \psi_w)
\]
(2.1.9)

from equation (2), (14), (21) and (25) - (2\theta_u + \phi_v + \psi_w) - 2\tau_t - 2\xi_x = 0 which gives
\[
\theta_u = -\frac{1}{2} (\phi_v + \psi_w) - \tau_t - \xi_x
\]
(2.1.10)

This implies
\[
\theta = \left[ -\frac{1}{2} (\phi_v + \psi_w) - \tau_t - \xi_x \right] u + A(t, x)
\]
(2.1.11)

from equation (2.1.11)
\[
\theta_t = A_x
\]
(2.1.12)

and
\[
\theta_t = A_t
\]
(2.1.13)

using equation (2.1.12) in equation (1)
\[
\phi = A_x - v (\xi_x + \tau_t)
\]
(2.1.14)

using equation (2.1.13) in equation (10)
\[
\psi = A_t - w (\xi_x + \tau_t)
\]
(2.1.15)

from equation (19) \(\theta_t - \phi_u + \psi_t = 0\), we have \(A_u + A_t = A_{xx}\)
(2.1.16)

where \(A(t, x)\) is an arbitrary function of \(x\) and \(t\) which is the solution of given equation (1) of wave propagation with dissipation, again from equation (2), (14), (21) and (25) we have
\[
\theta_u = \phi_v = \psi_w = c_3 \text{ (say)}
\]
(2.1.17)

which gives
\[
\tau = c_3 t + c_1
\]
(2.1.18)

\[
\xi = c_3 x + c_2
\]
(2.1.19)

from (2.1.14) using (2.1.17) and (2.1.18)
\[
\phi = A_x + 2 c_3 v
\]
(2.1.20)

from (2.1.15) using (2.1.18)
\[
\psi = A_t + 2 c_3 w
\]
(2.1.21)

from (2.1.11)
\[
\theta = c_3 u + A(t, x)
\]
(2.1.22)

hence the require vector fields
\[
X_1 = \partial_u, \quad X_2 = \partial_v, \quad X_3 = -t \partial_t - x \partial_x + u \partial_u + 2 v \partial_v + 2 w \partial_w,
\]
\[
X_\infty = A(t, x) \partial_u + A_x \partial_v + A_t \partial_w
\]
when \(v = 0\) and \(w = 0\) the commutation relation table given as follows

<table>
<thead>
<tr>
<th></th>
<th>X_1</th>
<th>X_2</th>
<th>X_3</th>
<th>X_\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>X_1</td>
<td>0</td>
<td>0</td>
<td>X_1</td>
<td>X_{A^-}</td>
</tr>
<tr>
<td>X_2</td>
<td>0</td>
<td>0</td>
<td>X_2</td>
<td>X_{A^-}</td>
</tr>
<tr>
<td>X_3</td>
<td>-X_1</td>
<td>-X_2</td>
<td>0</td>
<td>X_{-t A^- - X A^-}</td>
</tr>
<tr>
<td>X_\infty</td>
<td>-X_{A^-}</td>
<td>-X_{A^-}</td>
<td>- (X_{-t A^- - X A^-})</td>
<td>0</td>
</tr>
</tbody>
</table>
where \( x_A^\gamma = - A_t \partial_u, \ x_A^{\alpha \gamma} = - A_x \partial_u \)

Symmetry groups are
\[ G_1 : (t + \varepsilon, x, u), \ G_2 : (t, x + \varepsilon, u), \ G_3 : (t e^{-\varepsilon}, x e^{\varepsilon}, u e^{\varepsilon}), \ G_\infty : (t, x, u + \varepsilon A) \],

Solution for symmetry groups
\[ u^{(1)} = f(t - \varepsilon, x), \ u^{(2)} = f(t, x - \varepsilon), \ u^{(3)} = e^\varepsilon f(t e^\varepsilon, x e^\varepsilon), \ u^{\infty} = f(t, x) + \varepsilon A(t, x) \]

**Conclusion**

The proposed method has been successfully applied to analyzing the wave equation. Potential and Lie point symmetry have been obtained for the wave equation. Further, using Lie point symmetry groups, the solution of the problem have been obtained. One can simply write the differential equations as a set of second order equations and then the differential forms can be written by inspection. The method is also easy to apply for symbolic computation for Lie point symmetry, cf. Edelen [7]. Carminati et al.[1] introduces a useful computer program liesymm in MAPLE is use the proposed method. Thus, the proposed method has been extended to solve a large class of problems in nonlinear differential equations.

**REFERENCES**


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