p -Adic Invariant Integral on $\mathbb{Z}_p$ Associated with the Changhee's q -Bernoulli Polynomials

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Abstract. In this paper, we study some properties of Changhee's q -Bernoulli polynomials which are derived from p -adic invariant integral on $\mathbb{Z}_p$. By using these properties, we give some interesting identities related to higher-order q -Bernoulli polynomials.

Keywords: Changhee's q -Bernoulli polynomials, p -adic invariant integral, higher-order q -Bernoulli polynomials

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of p -adic integers, the field of p -adic numbers, the complex number field and the

completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$. When one talks of q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < p^{-1/p-1}$, so that $g^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. We use the following notation in this paper:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \quad (\text{see [8] - [16]}).$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let d be a fixed integer and let p be a fixed prime number. We set

$$\mathbb{X}_d = \lim_N \mathbb{Z}/dp^N\mathbb{Z}, \quad \mathbb{X}^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [1]-[11]). Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = \int_{\mathbb{X}} f(x) d\mu_0(x). \quad (1.1)$$

Thus, by (1.1), we see

$$I_0(f_1) = I_0(f) + f'(0), \quad (1.2)$$

where $f_1(x) = f(x+1)$. From (1.2), we can derive

$$I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i), \quad (\text{see [9], [10], [14], [15]}) \quad (1.3)$$

where $f_n(x) = f(x+n)$, $f'(i) = \frac{df(x)}{dx}|_{x=i}$.

As is well known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{Z}_{\geq 0}) \quad (1.4)$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the n -th Bernoulli numbers of order r (see [1]-[16]). By (1.2), we easily set

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1 + \cdots + x_r)t} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{t^n}{n!}. \quad (1.5)$$

From (1.5), we have

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (x_1 + \cdots + x_r)^n d\mu_0(x_1) \cdots d\mu_0(x_r) = B_n^{(r)}, (n \geq 0), \quad (1.6)$$

and

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} (x_1 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r) = B_n^{(r)}(x), (n \geq 0). \quad (1.7)$$

In this paper, we consider Changhee q -Bernoulli number and polynomials which are derived from multivariate p -adic invariant integral on $\mathbb{Z}_p$. and investigate some properties of Changhee q -Bernoulli polynomials.

2. Changhee q -Bernoulli polynomials

Let $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ be positive integers. For $w \in \mathbb{Z}_p$, we consider the modified Changhee q -Bernoulli polynomials which are derived from multivariate p -adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} & B_{n,q}^{(k)}(w|a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k) \\ &= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{b_1 x_1 + \cdots + b_k x_k} [w + a_1 x_1 + \cdots + a_k x_k]_q^n d\mu_0(x_1) \cdots d\mu_0(x_k), \quad (2.1) \end{aligned}$$

and

$$\begin{aligned} & B_{n,q}^{(k)}(a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{b_1 x_1 + \dots + b_k x_k} [a_1 x_1 + \dots + a_k x_k]_q^n d\mu_0(x_1) \dots d\mu_0(x_k). \end{aligned} \quad (2.2)$$

From (1.1) and (2.1), we have

$$\begin{aligned} & B_{n,q}^{(k)}(w | a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k) \\ &= \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{b_1 x_1 + \dots + b_k x_k} [w + a_1 x_1 + \dots + a_k x_k]_q^n d\mu_0(x_1) \dots d\mu_0(x_k) \\ &= \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \lim_{N \rightarrow \infty} \left(\frac{1}{P^N} \right)^k \\ &\quad \times \sum_{x_1, x_2, \dots, x_k=0}^{P^N-1} q^{r a_1 x_1 + \dots + r a_k x_k + b_1 x_1 + \dots + b_k x_k} \\ &= \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \left(\frac{\log q}{q-1} \right)^k \prod_{j=1}^k \left(\frac{r a_j + b_j}{[r a_j + b_j]_q} \right) \end{aligned} \quad (2.3)$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. For $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \in \mathbb{Z}_{\geq 0}$ and $w \in \mathbb{Z}_p$, we have

$$\begin{aligned} & B_{n,q}^{(k)}(w | a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k) \\ &= \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \prod_{j=1}^k \left(\frac{r a_j + b_j}{[r a_j + b_j]_q} \right), \end{aligned}$$

where $n \geq 0, k \geq 1$.

Let us take $b_1 = 1, b_2 = 2, \dots, b_k = k$. Then, by theorem 2.1, we get

$$\begin{aligned} & B_{n,q}^{(k)}(w | \underbrace{1, 1, \dots, 1}_{k\text{-times}}; 1, 2, \dots, k) \\ &= \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \prod_{j=1}^k \left(\frac{r+j}{[r+j]_q} \right) \\ &= \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \frac{\binom{r+k}{k}}{\binom{r+k}{k}_q} \frac{k!}{[k]_q!}, \end{aligned} \quad (2.4)$$

where $\binom{r}{k}_q = \frac{[r]_q \cdots [r-k+1]_q}{[k]_q!}$, $[k]_q! = [k]_q [k-1]_q \cdots [2]_q [1]_q$. Therefore, by (2.4), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0, k \geq 1$, we have*

$$B_{n,q}^{(k)}(w \mid \underbrace{1, 1, \dots, 1}_{k\text{-times}}; 1, 2, \dots, k) = \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \frac{\binom{r+k}{k}_q}{\binom{r+k}{k}_q} \frac{k!}{[k]_q!}.$$

The multiple Barnes' Bernoulli polynomials are defined by the generating function to be

$$\prod_{j=1}^k \left(\frac{w_j}{e^{w_j t} - 1} \right) t^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x|w_1, \dots, w_k) \frac{t^n}{n!}, \quad (2.5)$$

where $w_i > 0, 0 < t < 1$ (see [8],[9],[15])). The numbers $B_n^{(k)}(0|w_1, w_2, \dots, w_k) = B_n^{(k)}(w_1, \dots, w_k)$ are called the Barnes' Bernoulli numbers of order k . If we take $x = rx, w_k = r+k$ and $t = \log q$ in (2.5), then we have

$$\frac{(r+1) \cdots (r+k)}{(q^{r+1} - 1) \cdots (q^{r+k} - 1)} q^{rx} = \sum_{s=0}^{\infty} B_s^{(k)}(rx|r+1, \dots, r+k) \frac{(\log q)^s}{s!}.$$

Note that

$$\lim_{q \rightarrow 1} B_{n,q}^{(k)}(w|a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k) = B_n^{(k)}(a_1, \dots, a_k)$$

For $k = 1$ and $w = 0$, let $B_{n,q} = B_{n,q}^{(1)}(0|1)$. Then we have

$$B_{n,q} = B_{n,q}^{(1)}(0|1) = \left(\frac{\log q}{1-q} \right) n \sum_{m=0}^{\infty} q^m [m]_q^{n-1} - \log q \sum_{m=0}^{\infty} q^m [m]_q^n.$$

From (2.1), we can derive

$$\begin{aligned}
 & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{b_1 x_1 + \cdots + b_k x_k} [a_1 x_1 + \cdots + a_k x_k]_q^n d\mu_0(x_1) \cdots d\mu_0(x_k) \\
 &= (q-1) \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{(b_1 - a_1)x_1 + \cdots + (b_k - a_k)x_k} \\
 &\quad \times [a_1 x_1 + \cdots + a_k x_k]_q^{n+1} d\mu_0(x_1) \cdots d\mu_0(x_k) \tag{2.6} \\
 &+ \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} q^{\sum_{j=1}^k (b_j - a_j)x_j} [a_1 x_1 + \cdots + a_k x_k]_q^n d\mu_0(x_1) \cdots d\mu_0(x_k) \\
 &= (q-1)B_{n+1,q}^{(k)}(a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k) \\
 &\quad + B_{n,q}^{(k)}(a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k).
 \end{aligned}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.3. For $n \geq 0, k \geq 1$, we have

$$\begin{aligned}
 B_{n,q}^{(k)}(a_1, \dots, a_k; b_1, \dots, b_k) &= (q-1)B_{n+1,q}^{(k)}(a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k) \\
 &\quad + B_{n,q}^{(k)}(a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k).
 \end{aligned}$$

It is easy to show that

$$\begin{aligned}
 & \sum_{j=0}^i \binom{i}{j} (q-1)^j \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [a_1 x_1 + \cdots + a_k x_k]_q^{n-i+j} \\
 &\quad \times q^{b_1 x_1 + \cdots + b_k x_k} d\mu_0(x_1) \cdots d\mu_0(x_k) \\
 &= \sum_{j=0}^{i-1} \binom{i-1}{j} (q-1)^j \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [a_1 x_1 + \cdots + a_k x_k]_q^{n-i+j} \\
 &\quad \times q^{(b_1 + a_1)x_1 + \cdots + (b_k + a_k)x_k} d\mu_0(x_1) \cdots d\mu_0(x_k) \\
 &= \sum_{j=0}^{i-1} \binom{i-1}{j} (q-1)^j B_{n-i+j,q}^{(k)}(a_1, \dots, a_k; b_1 + a_1, \dots, b_k + a_k). \tag{2.7}
 \end{aligned}$$

Therefore, by (2.7), we obtain the following theorem

Theorem 2.4. For $i \geq 1, n \geq 0, k \geq 1$, we have

$$\begin{aligned} & \sum_{j=0}^i \binom{i}{j} (q-1)^j B_{n-i+j,q}^{(k)}(a_1, \dots, a_k; b_1, \dots, b_k) \\ &= \sum_{j=0}^{i-1} \binom{i-1}{j} (q-1)^j B_{n-i+j,q}^{(k)}(a_1, \dots, a_k; b_1 + a_1, \dots, b_k + a_k). \end{aligned}$$

In the special case, $k = 1$, we have

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} (q-1)^j B_{j,q}^{(1)}(a_1, b_1) &= \sum_{j=0}^n \binom{n}{j} (q-1)^j \int_{\mathbb{Z}_p} [a_1 x]_q^j q^{b_1 x} d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} q^{(na_1 + b_1)x} d\mu_0(x) \\ &= \left(\frac{q-1}{\log q} \right) \frac{na_1 + b_1}{[na_1 + b_1]}. \end{aligned}$$

From (2.1) and (2.3), we have

$$\begin{aligned} & B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k) \\ &= \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \prod_{j=1}^k \left(\frac{ra_j + b_j}{[ra_j + b_j]_q} \right) \\ &= \sum_{i=0}^n \binom{n}{i} [w]_q^{n-i} q^{wi} B_{i,q}^{(k)}(a_1, \dots, a_k; b_1, \dots, b_k) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
& B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k) \\
&= \underbrace{\int_{\mathbb{X}_d} \dots \int_{\mathbb{X}_d}}_{k\text{-times}} q^{b_1 x_1 + \dots + b_k x_k} [a_1 x_1 + \dots + a_k x_k]_q^n d\mu_0(x_1) \dots d\mu_0(x_k) \\
&= [l]_q^{n-k} \sum_{i_1, \dots, i_k=0}^{l-1} q^{b_1 i_1 + \dots + b_k i_k} \\
&\quad \times \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \left[\frac{w + a_1 i_1 + \dots + a_k i_k}{l} + a_1 x_1 + \dots + a_k x_k \right]_{ql}^n \quad (2.9) \\
&\quad \times q^{lb_1 x_1 + \dots + lb_k x_k} d\mu_0(x_1) \dots d\mu_0(x_k) \\
&= [l]_q^{n-k} \sum_{i_1, \dots, i_k=0}^{l-1} q^{b_1 i_1 + \dots + b_k i_k} \\
&\quad \times B_{n,ql}^{(k)} \left(\frac{w + a_1 i_1 + \dots + a_k i_k}{l} \mid a_1, \dots, a_k; b_1, \dots, b_k \right)
\end{aligned}$$

(Distribution relation for Changhee q -Bernoulli polynomials) An obvious generating function $F_q(w, t)$ of $B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k)$ is obtained from Theorem 2.1 as follows;

$$\begin{aligned}
F_q(w, t) &= \left(\frac{\log q}{q-1} \right)^k \sum_{n=0}^{\infty} (q-1)^n B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k) \frac{t^n}{n!} \\
&= \left(\frac{\log q}{q-1} \right)^k e^{-t} \sum_{i=0}^{\infty} \left(\prod_{j=1}^k \frac{ia_j + b_j}{[ia_j + b_j]_q} \right) q^{wi} \frac{t^i}{i!}. \quad (2.10)
\end{aligned}$$

Note that

$$\lim_{q \rightarrow 1} F_q(w, t) = \frac{a_1 a_2 \dots a_k t^k}{(e^{a_1 t} - 1) \dots (e^{a_k t} - 1)}.$$

Differentiating both sides with respect to t in (2.10) and comparing coefficients, we have

$$\begin{aligned}
& q^w B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k) - B_{n,q}^{(k)}(w|a_1, a_2, \dots, a_k; b_1 - a_1, \dots, b_k - a_k) \\
&= (q-1) B_{n+1,q}^{(k)}(w|a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k). \quad (2.11)
\end{aligned}$$

It is easy to show that

$$\sum_{i=0}^n \binom{n}{i} (q-1)^i B_{i,q}^{(k)}(a_1, \dots, a_k; b_1, \dots, b_k) = \left(\frac{\log q}{q-1} \right)^k \prod_{j=1}^k \left(\frac{na_j + b_j}{[na_j + b_j]_q} \right). \quad (2.12)$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0, k \geq 1$, we have*

$$\begin{aligned} q^w B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1, \dots, b_k) - B_{n,q}^{(k)}(w|a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k) \\ = (q-1) B_{n+1,q}^{(k)}(w|a_1, \dots, a_k; b_1 - a_1, \dots, b_k - a_k) \end{aligned}$$

Moreover,

$$\sum_{i=0}^n \binom{n}{i} (q-1)^i B_{i,q}^{(k)}(a_1, \dots, a_k; b_1, \dots, b_k) = \left(\frac{\log q}{q-1} \right)^k \prod_{j=1}^k \left(\frac{na_j + b_j}{[na_j + b_j]_q} \right)$$

Let us assume that $a_1 = \dots = a_k = 1, b_1 = h, b_2 = h-1, \dots, b_k = h-k+1$, where $h \in \mathbb{Z}$. Then we have

$$\begin{aligned} B_{n,q}^{(k)}(w|\underbrace{1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1) \\ = \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \prod_{j=1}^k \left(\frac{r+h-j+1}{[r+h-j+1]_q} \right) \\ = \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \frac{\binom{r+h}{k}}{\binom{r+h}{k}_q} \frac{k!}{[k]_q!}. \end{aligned} \quad (2.13)$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}, k \geq 1, n \geq 0$, we have*

$$\begin{aligned} B_{n,q}^{(k)}(w|1, 1, \dots, 1; h, h-1, \dots, h-k+1) \\ = \left(\frac{\log q}{q-1} \right)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} (-q^w)^r \frac{\binom{r+h}{k}}{\binom{r+h}{k}_q} \frac{k!}{[k]_q!}. \end{aligned}$$

Let us use the notation of q -product as follows :

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}) = \prod_{i=0}^{n-1} (1-aq^i). \quad (2.14)$$

From (2.13), we have

$$\begin{aligned}
 & B_{n,q}^{(k)}(w|\underbrace{1, 1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1) \\
 &= (\log q)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} q^{wr} (-1)^{r+k} \frac{(r+h)_k}{(q^{r+h-k+1}; q)_k} \\
 &= (\log q)^k \frac{1}{(1-q)^n} \sum_{r=0}^n \binom{n}{r} q^{wr} (-1)^{r+k} k! \binom{r+h}{k} \\
 &\quad \times \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q q^{m(r+h-k+1)} \\
 &= k! (\log q)^k \frac{1}{(1-q)^n} \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q q^{m(h-k+1)} \\
 &\quad \times \sum_{r=0}^n \binom{n}{r} (-1)^{r+k} \binom{r+h}{k} q^{(m+w)r}.
 \end{aligned} \tag{2.15}$$

By (2.15), we get

$$\begin{aligned}
 & B_{n,q}^{(k)}(w|\underbrace{1, 1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1) \\
 &= \frac{(\log q)^k}{(1-q)^n} k! \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q q^{m(h-k+1)} \sum_{r=0}^n \binom{n}{r} \binom{r+h}{k} (-1)^{r+k} q^{(m+w)r}.
 \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0, k \geq 1$, we have*

$$\begin{aligned}
 & B_{n,q}^{(k)}(w|1, 1, \dots, 1; h, h-1, \dots, h-k+1) \\
 &= \frac{(\log q)^k}{(1-q)^m} k! \sum_{m=0}^{\infty} \binom{m+k-1}{m}_q q^{m(h-k+1)} \sum_{r=0}^n \binom{n}{r} \binom{r+h}{k} (-1)^{r+h} q^{(m+w)r}.
 \end{aligned} \tag{2.17}$$

Remark. From (2.15), we can derive

$$\begin{aligned}
 & q^r B_{n,q}^{(k)}(w+1 | \underbrace{1, 1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1) \\
 &= \frac{(\log q)^k}{(1-q)^n} q^h \sum_{j=0}^n \binom{n}{j} q^{(w+1)j} (-1)^{j+k} \frac{(j+h)_k}{(q^{j+h}; q^{-1})_k} \\
 &= \frac{(\log q)^k}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^{j+k} q^{wj} ([j+h]_q (q-1) + 1) \frac{(j+h)_k}{(q^{j+h}; q^{-1})_k} \\
 &= \frac{(q-1)(\log q)^k}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^{j+k} q^{wj} \frac{(j+h-1)_{k-1}}{(q^{j+h-1}; q^{-1})_{k-1}} (j+h) \\
 &+ \frac{(\log q)^k}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^{j+k} q^{wj} \frac{(j+h)_k}{(q^{j+h}; q^{-1})_k} \\
 &= h(1-q) \log q B_{n,q}^{(k-1)}(w | \underbrace{1, 1, \dots, 1}_{k\text{-times}}; h-1, h-2, \dots, h-k) \\
 &- n \log q B_{n-1,q}^{(k-1)}(w | \underbrace{1, 1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1) \\
 &+ B_{n,q}^{(k)}(w | \underbrace{1, 1, \dots, 1}_{k\text{-times}}; h, h-1, \dots, h-k+1).
 \end{aligned} \tag{2.18}$$

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