Coefficient Inequalities for Certain Subclass of $p-$Valent Functions of Complex Order

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Abstract

This paper introduces a new subclass of $p-$valent functions of complex order which is denoted by $S_p(b, \lambda, \alpha)$ with $0 \leq \lambda \leq 1$ and $\alpha > 1$. The coefficient inequality and Fekete-Szegö inequality for functions $f$ belongs to this class are obtained.

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1 Introduction

Let $A_p$ be the class of $p-$valent functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}$$

which are analytic in the open unit disc $D = \{z : |z| < 1\}$ and $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

In 2012, Sharma and Saroja introduced some subclasses of $p-$valent function of complex order and obtained the coefficient inequality and Fekete-Szegö inequality for functions in these classes.

Definition 1.1 ([3]) Let $b$ be a non-zero complex number and $\alpha > 1$. A function $f(z)$ of the form (1) is said to be in the class $M_p(b, \alpha)$ if
\[ \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] < \alpha, \quad z \in D. \]

It is noted that
\[ M_p(1, \alpha) = M_p(\alpha) \] defined by Srivastava and Owa in [4]
\[ M_1(1, \alpha) = M(\alpha) \] defined by Owa and Nishiwaki in [1]

**Definition 1.2** ([3]) Let \( b \) be a non-zero complex number and \( \alpha > 1 \). A function \( f(z) \) of the form (1) is said to be in the class \( N_p(b, \alpha) \) if
\[ \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right] < \alpha, \quad z \in D. \]

It is noted that
\[ N_p(1, \alpha) = N_p(\alpha) \] defined by Srivastava and Owa in [4]
\[ N_1(1, \alpha) = N(\alpha) \] defined by Owa and Nishiwaki in [1]

In this paper, we define a subclass of \( p \)-valent functions of complex order. We determine the coefficient inequality and Fekete-Szegö inequality for functions \( f \) belongs to this class.

**Definition 1.3** Let \( b \) be a non-zero complex number and \( \alpha > 1 \). A function \( f(z) \) of the form (1) is said to be in the class \( S_p(b, \lambda, \alpha) \) if and only if
\[ \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z f'(z) + (1 - \lambda) f(z)} - 1 \right) \right] < \alpha, \quad 0 \leq \lambda \leq 1, \quad z \in D. \tag{2} \]

It is noted that
\[ S_p(b, 0, \alpha) = M_p(b, \alpha) \] and \( S_p(b, 1, \alpha) = N_p(b, \alpha) \) defined by Sharma and Saroja in [3]
\[ S_p(1, 0, \alpha) = M_p(\alpha) \] and \( S_p(1, 1, \alpha) = N_p(\alpha) \) defined by Srivastava and Owa in [4]

## 2 Preliminary Results

To prove our results, we require the following lemma.

**Lemma 2.1** ([2]) If \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is a function with positive real part and \( p(0) = 1 \) then for any complex number \( v \), we have
\[ |c_2 - vc_1^2| \leq 2\text{Max}\{1, |2v - 1|\} \]

The result is sharp for the functions
\[ p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}. \]
3 Main Results

The coefficient inequality for functions \( f \in S_p(b, \lambda, \alpha) \) is given by the following theorem.

**Theorem 3.1** Let \( f(z) \in S_p(b, \lambda, \alpha), 0 \leq \lambda \leq 1, \alpha > 1 \) and be given by (1) then

\[
|a_{p+n}| \leq \frac{\lambda p + (1 - \lambda)}{n!(\lambda(p + n) + (1 - \lambda))} \prod_{j=0}^{n-1} [2[b((\alpha - 1) + (p - 1)] + j] 
\]

(3)

**Proof:** Since \( f(z) \in S_p(b, \lambda, \alpha) \) then from the Definition 1.3, we have

\[
\text{Re} \left[ 1 + \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda zf'(z) + (1 - \lambda)f(z)} - 1 \right] < \alpha.
\]

Define a function \( p(z) \) such that

\[
p(z) = \frac{\alpha - \left(1 + \frac{1}{b} \left[ \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda zf'(z) + (1 - \lambda)f(z)} - 1 \right] \right)}{\alpha - (1 + \frac{1}{b}(p - 1))} = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D
\]

(4)

Here \( p(z) \) is a function with positive real part with \( p(0) = 1 \).

Replacing \( f(z), zf'(z), z^2 f''(z) \) with their equivalent expressions on both sides, we get

\[
\left[ 1 + \sum_{n=1}^{\infty} c_n z^n \right] [b(\alpha - 1) - (p - 1)] \left[ z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \right]
\]

\[
= [b(\alpha - 1) + 1] \left[ z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \right] - \left[ p z^p + \sum_{n=1}^{\infty} (p + n) a_{p+n} z^{p+n} \right]
\]

(5)

Comparing the coefficient of \( z^{p+n} \) on both sides of equations (5), we get

\[
-n[\lambda(p + n) + (1 - \lambda)]a_{p+n}
\]

\[
= [b(\alpha - 1) - (p - 1)] \left[ \lambda p + (1 - \lambda)]c_n + [\lambda(p + 1) + (1 - \lambda)]a_{p+1}c_{n-1} \right.
\]

\[
+ [\lambda(p + 2) + (1 - \lambda)]a_{p+2}c_{n-2} + \cdots + [\lambda(p + n - 2) + (1 - \lambda)]a_{p+n-2}c_2
\]

\[
+ [\lambda(p + n - 1) + (1 - \lambda)]a_{p+n-1}c_1 \]

(6)
Taking modulus on both sides of (6) and applying $|c_n| \leq 2 \ \forall n \geq 1$, we get

$$|a_{p+n}| \leq 2 \left[ \frac{|b|}{n} \frac{(\alpha - 1) + (p - 1)}{(\lambda (p + n) + (1 - \lambda))} \right] \left[ (\lambda p + (1 - \lambda)) + (\lambda (p + 1) + (1 - \lambda)) |a_{p+1}| \right.$$  

$$+ (\lambda (p + 2) + (1 - \lambda)) |a_{p+2}| + \cdots + (\lambda (p + n - 2) + (1 - \lambda)) |a_{p+n-2}|$$  

$$+ (\lambda (p + n - 1) + (1 - \lambda)) |a_{p+n-1}| \right] \tag{7}$$

For $n = 1$

$$|a_{p+1}| \leq \frac{\lambda p + (1 - \lambda)}{\lambda (p + 1) + (1 - \lambda)} \left[ 2 |b| (\alpha - 1) + (p - 1) \right]$$

Thus the result (3) is true for $n = 1$.

For $n = 2$

$$|a_{p+2}| \leq \frac{\lambda p + (1 - \lambda)}{2! [\lambda (p + 2) + (1 - \lambda)]} \prod_{j=0}^{1} \left[ 2 |b| (\alpha - 1) + (p - 1) + j \right]$$

Thus the result (3) holds true for $n=2$.

Suppose the result (3) is true for $n=k$

Now for $n=k+1$, we have

$$|a_{p+k+1}| \leq \left[ \frac{\lambda p + (1 - \lambda)}{\lambda (p + k + 1) + (1 - \lambda)} \right] \left[ 2 \left[ \frac{|b| (\alpha - 1) + (p - 1)}{(k + 1)} \right] \left[ 1 + 2 |b| (\alpha - 1) + (p - 1) \right] + \cdots \right.$$  

$$+ \frac{1}{k!} \prod_{j=0}^{k-1} \left[ 2 |b| (\alpha - 1) + (p - 1) + j \right]$$
\[ |a_{p+k+1}| \leq \frac{\lambda p + (1 - \lambda)}{(k+1)!\lambda(p+k+1) + (1-\lambda)} \prod_{j=0}^{k} \left[ 2|b|\alpha + (p-1) \right] + j\]

Thus the result (3) is true for \( n = k + 1 \).

By mathematical induction the result (3) is true for all values of \( n \). This completes the proof of Theorem 3.1.

Next, the result of Fekete-Szegö inequality is given by following theorem.

**Theorem 3.2** If \( f(z) \in S_p(b, \lambda, \alpha) \) then for any complex number \( \mu \)

\[ |a_{p+2} - \mu a_{p+1}^2| \leq \frac{\lambda p + (1 - \lambda)|b(\alpha - 1) + (p - 1)|}{\lambda(p + 2) + (1 - \lambda)} \]

\[ \max \left\{ 1, \left| 2\left[ b(1 - \alpha) + (p - 1) \right] \left( 2\mu(\lambda p + (1 - \lambda)) \right) \left( \frac{\lambda(p + 2) + (1 - \lambda)}{\lambda(p + 1) + (1 - \lambda)^2} - 1 \right) - 1 \right| \} \]

The result obtained is sharp.

**Proof**: Since \( f(z) \in S_p(b, \lambda, \alpha) \) then from (6), we obtain

\[ a_{p+1} = \frac{\lambda p + (1 - \lambda)|b(1 - \alpha) + (p - 1)|c_1}{\lambda(p + 1) + (1 - \lambda)} \]

and

\[ a_{p+2} = \frac{\lambda p + (1 - \lambda)|b(1 - \alpha) + (p - 1)|c_2}{2[\lambda(p + 2) + (1 - \lambda)]} \left[ c_2 + [b(1 - \alpha) + (p - 1)]c_1^2 \right] \]

For any complex number \( \mu \)

\[ a_{p+2} - \mu a_{p+1}^2 = \frac{\lambda p + (1 - \lambda)|b(1 - \alpha) + (p - 1)|c_2}{2[\lambda(p + 2) + (1 - \lambda)]} \left[ c_2 + [b(1 - \alpha) + (p - 1)]c_1^2 \right] - \mu \left( \frac{[b(1 - \alpha) + (p - 1)]^2\lambda(p) + (1 - \lambda)^2c_1^2}{[\lambda(p + 1) + (1 - \lambda)^2} \right) \]
\[
\begin{align*}
&= \frac{[\lambda p + (1 - \lambda)][b(1 - \alpha) + (p - 1)]}{2[\lambda(p + 2) + (1 - \lambda)]} \left( c_2 - [b(1 - \alpha) + (p - 1)] \left[ 2\mu[\lambda p + (1 - \lambda)] \right]
\right.
\left. \left( \frac{\lambda(p + 2) + (1 - \lambda)}{[\lambda(p + 1) + (1 - \lambda)]^2} - 1 \right)c_1^2 \right)
\]
\]
\[
= \frac{[\lambda p + (1 - \lambda)][b(1 - \alpha) + (p - 1)]}{2[\lambda(p + 2) + (1 - \lambda)]} \left[ c_2 - vc_1^2 \right]
\]

where
\[
v = [b(1 - \alpha) + (p - 1)] \left[ 2\mu[\lambda p + (1 - \lambda)] \left( \frac{\lambda(p + 2) + (1 - \lambda)}{[\lambda(p + 1) + (1 - \lambda)]^2} - 1 \right) \right]
\]

Taking modulus on both sides and applying Lemma 2.1, we get
\[
|a_{p+2} - \mu a_{p+1}^2| \leq \left| \frac{[\lambda p + (1 - \lambda)][b(1 - \alpha) + (p - 1)]}{2[\lambda(p + 2) + (1 - \lambda)]} \right| \left| c_2 - vc_1^2 \right|
\]
\[
\leq \frac{[\lambda p + (1 - \lambda)][|b|\alpha + (p - 1)]}{\lambda(p + 2) + (1 - \lambda)} \text{max} \{1, \left| 2v - 1 \right| \}
\]
\[
\leq \frac{[\lambda p + (1 - \lambda)][|b|\alpha + (p - 1)]}{\lambda(p + 2) + (1 - \lambda)} \text{max} \left\{ 1, \left| 2[b(1 - \alpha) + (p - 1)] \left[ 2\mu[\lambda p + (1 - \lambda)] \left( \frac{\lambda(p + 2) + (1 - \lambda)}{[\lambda(p + 1) + (1 - \lambda)]^2} - 1 \right) \right] - 1 \right| - 1 \right\}
\]

This proves Theorem 3.2. The result is sharp.

\[
|a_{p+2} - \mu a_{p+1}^2| = \frac{[\lambda p + (1 - \lambda)][|b|\alpha + (p - 1)]}{\lambda(p + 2) + (1 - \lambda)} \quad \text{if} \quad p(z) = \frac{1 + z^2}{1 - z^2}
\]

and

\[
|a_{p+2} - \mu a_{p+1}^2| = \frac{[\lambda p + (1 - \lambda)][|b|\alpha + (p - 1)]}{\lambda(p + 2) + (1 - \lambda)} \left| 2[b(1 - \alpha) + (p - 1)] \right| \left[ 2\mu[\lambda p + (1 - \lambda)] \left( \frac{\lambda(p + 2) + (1 - \lambda)}{[\lambda(p + 1) + (1 - \lambda)]^2} - 1 \right) \right] - 1 \quad \text{if} \quad p(z) = \frac{1 + z}{1 - z}
\]
By taking $\lambda = 0$ in Theorem 3.1 and Theorem 3.2, we obtain the following corollaries respectively

**Corollary 3.1** ([3]) If $f(z) \in M_p(b, \alpha)$ then

$$|a_{n+p}| \leq \frac{1}{n!} \prod_{j=0}^{n-1} [2|b|(|\alpha - 1| + (p - 1)) + j]$$

**Corollary 3.2** ([3]) If $f(z) \in M_p(b, \alpha)$ then for any complex number $\mu$, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq |b|(|\alpha - 1| + (p - 1)) \max\{1, 2[b(1 - \alpha) + (p - 1)]2|\mu - 1| - 1\}$$

The result is sharp.

By taking $\lambda = 1$ in Theorem 3.1 and Theorem 3.2, we obtain the following corollaries respectively

**Corollary 3.3** ([3]) If $f(z) \in N_p(b, \alpha)$ then

$$|a_{n+p}| \leq \frac{p}{n!(n+p)} \prod_{j=0}^{n-1} [2|b|(|\alpha - 1| + (p - 1)) + j]$$

**Corollary 3.4** ([3]) If $f(z) \in N_p(b, \alpha)$ then for any complex number $\mu$, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p}{(p+2)} [b(|\alpha - 1| + (p - 1)) \max\{1, 2[b(1 - \alpha) + (p - 1)] \left[2\mu p \left(\frac{p+2}{(p+1)^2}\right) - 1\right] - 1\}]$$

The result is sharp.

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**References**


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