A Property for the $\alpha$-Diagonally Dominant Matrix with Applications$^1$

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Abstract

In this paper, we present a new property for the $\alpha$ diagonally dominant matrix. As applications, we give some criteria to distinguish the nosingular H-matrix.

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1. Introduction

The $H$-matrix is a special class of matrices that arises in the applications of engineering and mathematical sciences. It plays an important role in numerical algebra, computational mathematics, mathematical physics, control theory, electric systems, economic mathematics and elastic dynamics and so on (see [4]). But it is difficult to determine whether a matrix is or not a nonsingular $H$-matrix in reality. Several criteria for verifying nonsingular $H$-matrix under certain conditions can be found in the literature [1, 2, 3, 5, 6], and some iterative algorithms [7, 8, 9] have been provided for identifying nonsingular $H$-matrices, but most of these algorithms are designed for implementing on a sequential computer, and they may ineffective for large scalar matrices. In this paper, we present several simple criteria for nonsingular $H$-matrix.

2. Notations and Definitions

Let $C^{n \times n}(R^{n \times n})$ be the set of complex (real) $n \times n$ matrices, $A = (a_{ij}) \in C^{n \times n}$, $N = \{1, 2, \ldots, n\}$. For any $i, j \in N$, we denote

$$
\alpha_i = \sum_{j \in N_1, j \neq i} |a_{ij}|, \beta_i = \sum_{j \in N_2, j \neq i} |a_{ij}|, \bar{\alpha}_i = \sum_{j \in N_1, j \neq i} |a_{ji}|, \bar{\beta}_i = \sum_{j \in N_2, j \neq i} |a_{ji}|,
$$

$$
\Lambda_i(A) = \sum_{j \neq i} |a_{ij}| = \alpha_i + \beta_i, S_i(A) = \sum_{j \neq i} |a_{ji}| = \bar{\alpha}_i + \bar{\beta}_i,
$$

where $N = N_1 \oplus N_2$, $N_1 = \{i \mid |a_{ii}| > \Lambda_i(A), i \in N\}$, $N_2 = \{i \mid 0 < |a_{ii}| \leq \Lambda_i(A), i \in N\}$. Let $A^{(k)} = (a^{(k)}_{ij})$ be the matrix that is $A$ after $k$th steps gauss elimination.

**Definition 1.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a diagonally dominant matrix if $|a_{ii}| \geq \Lambda_i(A)$ for all $i \in N$. $A = (a_{ij}) \in C^{n \times n}$ is said to be a strictly diagonally dominant matrix, denotes by $A \in SD$, if $|a_{ii}| > \Lambda_i(A)$ for all $i \in N$. If there exists a positive diagonal matrix $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, such that $AD = B \in SD$, then we call $A$ is a generalized strictly diagonally dominant matrix, denotes by $A \in GSD$.

**Definition 2.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a $\alpha_1$ diagonally dominant matrix, denotes by $A \in D(\alpha_1)$, if there exists $\alpha \in [0, 1]$ such that

$$
|a_{ii}| \geq \Lambda_i(A)^{\alpha} S_i(A)^{1-\alpha} \quad (1)
$$

for all $i \in N$. $A$ is said to be a strictly $\alpha_1$ diagonally dominant matrix, denotes by $A \in SD(\alpha_1)$, if inequality (1) is strict. If there exists a positive diagonal
matrix $D$, such that $AD = B \in SD(\alpha_1)$, then we call $A$ is a generalized strictly $\alpha_1$ diagonally dominant matrix and we denote by $A \in GSD(\alpha_1)$.

**Definition 3.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a $\alpha_2$ diagonally dominant matrix, denotes by $A \in D(\alpha_2)$, if there exists $\alpha \in [0, 1]$ such that

$$|a_{ii}| \geq \alpha \Lambda_i(A) + (1 - \alpha)S_i(A)$$  \hspace{1cm} (2)

for all $i \in N$. $A$ is said to be a strictly $\alpha_2$ diagonally dominant matrix, denotes by $A \in SD(\alpha_2)$, if inequality (2) is strict. If there exists a positive diagonal matrix $D$, such that $AD = B \in SD(\alpha_2)$, then we call $A$ is a generalized strictly $\alpha_2$ diagonally dominant matrix and we denote by $A \in GSD(\alpha_2)$.

**Definition 4.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a $\alpha_1$ double diagonally dominant matrix, denotes by $A \in DD(\alpha_1)$, if there exists $\alpha \in [0, 1]$ such that

$$|a_{i-i}a_{jj}| \geq (\Lambda_i(A)\Lambda_j(A))\alpha(S_i(A)S_j(A))^{1-\alpha}$$  \hspace{1cm} (3)

for all $i, j \in N$ with $i \neq j$. If inequality (3) is strict, then we call $A$ is a strictly $\alpha_1$ double diagonally dominant matrix and we denote by $A \in SDD(\alpha_1)$. If there exists a positive diagonal matrix $D$, such that $AD = B \in SDD(\alpha_1)$, then $A$ is said to be a generalized strictly $\alpha_1$ double diagonally dominant matrix and we denote by $A \in GSSDD(\alpha_1)$.

**Definition 5.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a $\alpha_2$ double diagonally dominant matrix, denotes by $A \in DD(\alpha_2)$, if there exists $\alpha \in [0, 1]$ such that

$$|a_{i-i}a_{jj}| \geq (\alpha \Lambda_i(A) + (1 - \alpha)S_i(A))(\alpha \Lambda_j(A) + (1 - \alpha)S_j(A))$$  \hspace{1cm} (4)

for all $i, j \in N$ with $i \neq j$. If inequality (4) is strict, then we call $A$ is a strictly $\alpha_2$ double diagonally dominant matrix and we denote by $A \in SDD(\alpha_2)$. If there exists a positive diagonal matrix $D$, such that $AD = B \in SDD(\alpha_2)$, then $A$ is said to be a generalized strictly $\alpha_2$ double diagonally dominant matrix and we denote by $A \in GSSDD(\alpha_2)$.

**Definition 6.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a $\alpha_3$ double diagonally dominant matrix, denotes by $A \in DD(\alpha_3)$, if there exists $\alpha \in [0, 1]$ such that

$$|a_{i-i}a_{jj}| \geq \alpha \Lambda_i(A)\Lambda_j(A) + (1 - \alpha)S_i(A)S_j(A)$$  \hspace{1cm} (5)

for all $i, j \in N$ with $i \neq j$. If inequality (5) is strict, then we call $A$ is a strictly $\alpha_3$ double diagonally dominant matrix and we denote by $A \in SDD(\alpha_3)$. If there exists a positive diagonal matrix $D$, such that $AD = B \in SDD(\alpha_3)$, then $A$ is said to be a generalized strictly $\alpha_3$ double diagonally dominant matrix and we denote by $A \in GSSDD(\alpha_3)$.

**Definition 7.** $A = (a_{ij}) \in C^{n \times n}$ is said to be a double diagonally dominant matrix if

$$|a_{i-i}a_{jj}| \geq \Lambda_i(A)\Lambda_j(A)$$  \hspace{1cm} (6)
for all \( i, j \in N \) with \( i \neq j \). If inequality (6) is strict, then we call \( A \) is a strictly double diagonally dominant matrix and we denote by \( A \in SDD \). If there exists a positive diagonal matrix \( D \), such that \( AD = B \in SDD \), then we call \( A \) is a generalized strictly double diagonally dominant matrix and we denote by \( A \in GSDD \).

**Definition 8.** Let \( A = (a_{ij}) \in C^{n \times n} \), then \( \mu(A) \) is said to be a comparison matrix of \( A \) if \( \mu(A) = (m_{ij}) \in R^{n \times n} \) with

\[
m_{ij} = \begin{cases} |a_{ii}|, i = j, \\ -|a_{ij}|, i \neq j. \end{cases}
\]

If the eigenvalues of \( \mu(A) \) have positive real parts, then we call \( \mu(A) \) is a nonsingular M-matrix and \( A \) is said to be a nonsingular H-matrix if and only if \( \mu(A) \) is a nonsingular M-matrix.

### 3. Main Results

In order to prove our main results, we need several lemmas, which we present in this section.

**Lemma 1** (see [1]). Let \( A = (a_{ij}) \in C^{n \times n} \), then \( \mu(A) \) is a nonsingular M-matrix and \( A \in GSDD \) if \( A \) satisfies one of the following conditions:

1. \( A \in SD(\alpha_1) \);
2. \( A \in SD(\alpha_2) \);
3. \( A \in SDD(\alpha_1) \);
4. \( A \in SDD(\alpha_2) \);
5. \( A \in SDD(\alpha_3) \).

**Lemma 2** (see [2]). If \( A = (a_{ij}) \in C^{n \times n} \in SD \), then \( A^{(n-1)} \in SD \).

**Lemma 3** (see [2]). If \( A = (a_{ij}) \in C^{n \times n} \) and \( A^{(n-1)} \notin GSDD \), then \( A \notin GSDD \).

**Lemma 4** (see [3]). If \( A = (a_{ij}) \in C^{n \times n} \) is a nonsingular H-matrix, then there exists at least one strict diagonally dominant row, namely \( N_1 \neq \Phi \).

**Lemma 5** (see [4]). \( A \) is an H-matrix if and only if \( A \in GSDD \).

**Lemma 6** (see [4]). If \( A = (a_{ij}) \in C^{n \times n} \) and \( B = \mu(A) + \mu(A)^T \in GSDD \), then \( \mu(A) \) is a nonsingular M-matrix and \( A \in GSDD \).

**Theorem 1.** Let \( A = (a_{ij}) \in C^{n \times n} \) with \( A = A^T \), then we have

1. If \( A \in SD(\alpha_1) \cup SD(\alpha_2) \), then \( A \in SD \);
2. If \( A \in SDD(\alpha_1) \cup SDD(\alpha_2) \cup SDD(\alpha_3) \), then \( A \in SDD \).

**Proof.** It follows from \( A = A^T \) for any \( i \in N \) that \( \Lambda_i(A) = S_i(A) \) and Theorem 1 is clear. \( \square \)

**Theorem 2.** Let \( A = (a_{ij}) \in C^{n \times n} \), then \( \mu(A) \) is a nonsingular M-matrix and \( A \in GSDD \) if \( A \) satisfies one of the following conditions:

1. \( A \in GSDD(\alpha_1) \);
2. \( A \in GSDD(\alpha_2) \);
3. \( A \in GSDD(\alpha_1) \);
4. \( A \in GSDD(\alpha_2) \);
5. \( A \in GSDD(\alpha_3) \).
**Proof.** From Definitions 2-6 and Lemma 1 we clearly see that there exists a positive diagonal matrix $D$, such that

$$AD = B \in SD(\alpha_1) \bigcup SD(\alpha_2) \bigcup SDD(\alpha_1) \bigcup SDD(\alpha_2) \bigcup SDD(\alpha_3).$$

Therefore, $\mu(B)$ is a nonsingular M-matrix and $AD = B \in GSD$, this leads to the conclusion that $\mu(A)$ is a nonsingular M-matrix and $A \in GSD$.

**Theorem 3.** If $A = (a_{ij}) \in C^{n\times n}$ satisfies one of the following conditions:
1. $A \in GSD(\alpha_1)$;
2. $A \in GSD(\alpha_2)$;
3. $A \in GSDD(\alpha_1)$;
4. $A \in GSDD(\alpha_2)$;
5. $A \in GSDD(\alpha_3)$.
Then $A^{(n-1)} \in GSD$.

**Proof.** It follows from Theorem 2 that $A \in GSD$. Therefore, there exists a positive diagonal matrix $D$ such that $AD = B \in SD$. Let $L^{-1}$ be the inverse Frobenius matrix

$$L^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-c_{21} & 1 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-c_{n1} & \cdots & \cdots & 1
\end{pmatrix}$$

with $c_{ii} = \frac{a_{ii}}{a_{ii}}$ $(i = 2, 3, \cdots, n)$, then $L^{-1}B = L^{-1}(AD) = (L^{-1}A)D = A^{(1)}D$. Then Lemma 2 leads to the conclusion that $A^{(1)}D \in SD$, which implies that $A^{(1)} \in GSD$. Therefore, $A^{(n-1)} \in GSD$ follows from the similar arguments.

**Theorem 4.** If $A = (a_{ij}) \in C^{n\times n}$ with $A^{(n-1)} \notin GSD$, then

$$A \notin SD(\alpha_1) \bigcup SD(\alpha_2) \bigcup SDD(\alpha_1) \bigcup SDD(\alpha_2) \bigcup SDD(\alpha_3).$$

**Proof.** It follows from $A^{(n-1)} \notin GSD$ and Lemma 3 we clearly see that $A \notin GSD$. If $A \in SD(\alpha_1) \bigcup SD(\alpha_2) \bigcup SDD(\alpha_1) \bigcup SDD(\alpha_2) \bigcup SDD(\alpha_3)$, then Lemma 1 implies that $A \in GSD$, which contradicts with the condition of Theorem 4 and the proof is completed.

**Theorem 5.** If $A = (a_{ij}) \in C^{n\times n}$ with $A^{(n-1)} \notin GSD$, then

$$A \notin GSD(\alpha_1) \bigcup GSD(\alpha_2) \bigcup GSDD(\alpha_1) \bigcup GSDD(\alpha_2) \bigcup GSDD(\alpha_3).$$

**Proof.** From $A^{(n-1)} \notin GSD$ and Lemma 3 we clearly see that $A \notin GSD$. If $A \in GSD(\alpha_1) \bigcup GSD(\alpha_2) \bigcup GSDD(\alpha_1) \bigcup GSDD(\alpha_2) \bigcup GSDD(\alpha_3)$, Then Theorem 2 leads to the conclusion that $A \in GSD$, which contradicts with the condition of Theorem 5 and the proof is completed.
Theorem 6. Let $A = (a_{ij}) \in C^{n \times n}$ and $B = \mu(A) + \mu(A)^T$, then $\mu(A)$ is a nonsingular M-matrix and $A \in GSD$ if $B$ satisfies one of the following conditions: (1) $B \in GSD(\alpha_1)$; (2) $B \in GSD(\alpha_2)$; (3) $B \in GSDD(\alpha_1)$; (4) $B \in GSDD(\alpha_2)$; (5) $B \in GSDD(\alpha_3)$.

Proof. It follows from Theorem 2 that $\mu(B)$ is a nonsingular M-matrix and $B \in GSD$, then Lemma 6 leads to the conclusion that $\mu(A)$ is a nonsingular M-matrix and $A \in GSD$.

References


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