Klein-Gordon Equation in Ideal Fluid

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Abstract

An ideal fluid traversed in all directions by straight vortex filament. In the present paper, we obtain the most general solution of the one-dimensional partial differential equation for the motion of an isolated stretched vortex filament combines both self induction and elasticity, in an ideal fluid, by computing the symmetry groups using the general prolongation formula for their infinitesimal generators of a groups of transformations. Several authors obtained solution of KG equation using different type of method [1], [2]. In recent year the authors Pelloni and Pinotsis[5], Plyukhin and Schofield[6], Rao and Kagoli[7] and Valery[9] worked for finding the solution of KG equation.

Keywords: Vortex filament, Translation, Hyperbolic Rotation

1. Introduction

1.1 The Klein-Gordon Equation in Ideal Fluid:

An ideal fluid traversed in all directions by straight vortex filaments, the motion of a isolated-stretched vortex filament combines both self induction and elasticity gives the KG (Klein-Gordon) equation in the standard form

\[ u_{tt} = c^2 u_{xx} - \left( m^2 c^4 / \hbar^2 \right) u \]  \hspace{1cm} (1.1.1)

Where \( u \) is wave function, \( c \) denotes the velocity of light, \( \hbar \) is plank constant and \( m \) is mass. The mass part of the KG equation describes the rotation of the helical
curve about the screw axis due to the hydrodynamics self-induction of the bent vortex filament (see Valery[8], page 5).

2. Main Results

The most general solution of KG equation

\[ u_{tt} = c^2 u_{xx} + \left( \frac{m^2 c^4}{h^2} \right) u \]  \hspace{1cm} (2.1)

in a ideal fluid for the motion of an isolated-stretched vortex filament combines with both self-induction and elasticity in the form which is the second order differential equation with two independent variables and one dependent variable, is obtained by computing the symmetry groups using the general prolongation formula for their infinitesimal generators of a group of transformations (see Olver[4], page 110). Let

\[ v = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u \]  \hspace{1cm} (2.2)

be a vector field of the KG equation (2.1) on \(X \times U\). Now, our aim is to calculate the coefficient functions \(\xi, \tau\) and \(\phi\) so that the corresponding one-parameter group \(exp(\varepsilon v)\) is a symmetry group of the KG equation. Using prolongation formula to determine the second prolongation of \(v\)

\[ pr^{(2)}v = v + \phi^x (\partial/\partial u_x) + \phi^t (\partial/\partial u_t) + \phi^{xx} (\partial/\partial u_{xx}) + \phi^{xt} (\partial/\partial u_{xt}) + \phi^{tt} (\partial/\partial u_{tt}) \]  \hspace{1cm} (2.3)

and infinitesimal criterion of invariance equation takes the form

\[ \phi'' - c^2 \phi^{xx} + \left( \frac{m^2 c^4}{h^2} \right) \phi = Q( u_{tt} - c^2 u_{xx} + \left( \frac{m^2 c^4}{h^2} \right) u ) \]  \hspace{1cm} (2.4)

in which \(Q(x, t, u^{(2)})\) depend up-to second order derivatives of \(u\). By substituting the values of \(\phi''\), \(\phi^{xx}\) and \(\phi\) in equation (2.4) and equating the coefficients of the terms in the first and second order partial derivatives of \(u\), the determining equations for the symmetry group of the one-dimensional KG equation in ideal fluid are found as follows (Table 1)
Table 1. The Determine Equation Table

<table>
<thead>
<tr>
<th>Monomial</th>
<th>Coefficient</th>
<th>Equation Number</th>
<th>Monomial</th>
<th>Coefficient</th>
<th>Equation Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_x u_x$</td>
<td>$-2\xi_u = 0$</td>
<td>1</td>
<td>$u_x^2 u_t$</td>
<td>$c^2 \tau_{uu} = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$2c^2 \tau_u = 0$</td>
<td>2</td>
<td>$u_t^3$</td>
<td>$-\tau_{uu} = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$-\xi_{uu} = 0$</td>
<td>3</td>
<td>$u_x^5$</td>
<td>$c^2 \xi_{uu} = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$c^2 \tau_u = 0$</td>
<td>4</td>
<td>$u_x u_t$</td>
<td>$-2\xi_{uu} + 2c^2 \tau_{xx} = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$-3\tau_u = 0$</td>
<td>5</td>
<td>$u_t^3$</td>
<td>$(\phi_{uu} - 2c^2 \tau_{xx}) = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$3c^2 \xi_u = 0$</td>
<td>6</td>
<td>$u_x^5$</td>
<td>$-(\phi_{uu} - 2\xi_{uu}) c^2 = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$-2\xi_t + 2c^2 \tau_t = 0$</td>
<td>7</td>
<td>$u_t$</td>
<td>$(2\phi_{uu} - \tau_x) + c^2 \tau_{xx} = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$(\phi_u - 2\tau_t) = Q$</td>
<td>8</td>
<td>$u_t$</td>
<td>$-\xi_{uu} - c^2 (2\phi_{uu} - \xi_{uu}) = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$u_x u_x$</td>
<td>$-c^2 (\phi_u - 2\tau_t) = -c^2 Q$</td>
<td>9</td>
<td>$u_t^3 u_x^2$</td>
<td>$c_4 \phi - c^2 \phi_{uu} + \frac{m^2 c^4}{\hbar^2} \phi - \frac{m^2 c^4}{\hbar^2} Qu = 0$</td>
<td>1</td>
</tr>
</tbody>
</table>

The requirement for equation (1) to (6) is that $\xi$ and $\tau$ are independent of $u$, equation (15) and (16) gives $\phi = \beta u + \alpha$ where $\alpha = \alpha(x, t)$ and $\beta = \beta(x, t)$ are functions. The equation (7), (8) and (9) gives $\xi_{uu} = c^2 \tau_{uu}$ and $\xi_{uu} = \tau_{uu}$ form the equation (17) and (18) we get $\beta_u = 0$ and $\beta_t = 0$, from (19) we find $\beta = Q = c_4 (\hbar^2 / m^2 c^4)$. Again form the equation (7), (8) and (9) we found $\xi = c_3 t + \pi(x)$, $\tau c^2 = c_3 x + 2 \alpha(t)$ and $\beta(t) = c_2$, $\pi(x) = c_1$. Since all the determining equations are satisfied then the coefficient function are $\xi = c_1 t + c_1$, $\tau c^2 = c_1 x + c_2$ and $\phi = c_4 (\hbar^2 / m^2 c^4)$ $u + \alpha$ where $c_1$, $c_2$, $c_3$ and $c_4$ are arbitrary constant and $\alpha$ is an arbitrary solution. The Lie algebras of infinitesimal symmetries of the KG equation in an ideal fluid is spanned by the four vector fields $v_1 = \partial_x$, $v_2 = (1/c^2) \partial_t$, $v_3 = t \partial_x + \frac{x}{c^2} \partial_t$, $v_4 = (\hbar^2 / m^2 c^4) u \partial_u$ and the infinite-dimensional sub-algebra $v_\alpha = \alpha \partial_u$. The commutation relation between these vector fields are as follows (see Table 2)
Table 2: The Commutation Relation Table

<table>
<thead>
<tr>
<th></th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>0</td>
<td>0</td>
<td>$v_2$</td>
<td>0</td>
<td>$v_\alpha$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>0</td>
<td>0</td>
<td>$v_1$</td>
<td>0</td>
<td>$(1/c^2)v_\alpha$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$-v_2$</td>
<td>$-v_1$</td>
<td>0</td>
<td>0</td>
<td>$v_{\alpha'}$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-(h^2/m^2c^4)v_\alpha$</td>
</tr>
<tr>
<td>$v_\alpha$</td>
<td>$-v_{\alpha'}$</td>
<td>$-(1/c^2)v_\alpha$</td>
<td>$-v_{\alpha'}$</td>
<td>$(h^2/m^2c^4)v_\alpha$</td>
<td>0</td>
</tr>
</tbody>
</table>

Where $\alpha' = t \alpha_0 + (x/c^2) \alpha$. Next, we obtained the one-parameter groups $G_i$ generated by the $v_i$ are as follows

$G_1: (x + \varepsilon, t, u)$, $G_2: (x, t + (\varepsilon/c^2), u)$, $G_3: (x \cosh(\varepsilon/c) + tc \sinh(\varepsilon/c), t \cosh(\varepsilon/c) + (x/c) \sinh(\varepsilon/c), u)$, $G_4: (x, t, e^{\varepsilon/c^2}(\varepsilon/c^2)} u)$, $G_\alpha: (x, t, u + \varepsilon \alpha)$ where each $G_i$ is a symmetry group. The solution of KG equation by using its different symmetry groups are given by

$u^{(1)} = f(x- \varepsilon, t), u^{(2)} = f(x, t- (\varepsilon/c^2))$, $u^{(3)} = f(x \cosh(\varepsilon/c) - tc \sinh(\varepsilon/c), t \cosh(\varepsilon/c) - (x/c) \sinh(\varepsilon/c))$, $u^{(4)} = e^{\varepsilon/c^2}(\varepsilon/c^2)} f(x, t), u^{(\alpha)} = f(x, t) + \varepsilon \alpha(x, t)$ where $u = f(x, t)$ be an assumed solution of KG equation, $\varepsilon$ is any real number and $\alpha(x, t)$ any other solution.

3 Conclusion

In our investigation the symmetry group $G_4$ and $G_\alpha$ reflects the linearity of the KG equation in ideal fluid. The group $G_1$ and $G_2$ are space and time invariance of the KG equation respectively. The group $G_3$ is well-known hyperbolic rotational symmetry group. At the end the most general solution that we can obtain from a given solution $u = f(x, t)$, by group transformations is in the form given below

$$u = e^{\varepsilon/c^2}(\varepsilon/c^2)} f \left( \begin{array}{c} x \cosh \left( \frac{\varepsilon_3}{c} \right) - tc \sinh \left( \frac{\varepsilon_3}{c} \right) - \varepsilon_1, \\ t \cosh \left( \frac{\varepsilon_3}{c} \right) - \left( \frac{x}{c} \right) \sinh \left( \frac{\varepsilon_3}{c} \right) - \varepsilon_2 \end{array} \right) + \alpha (x, t)$$

(3.1)

where $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon_4$ are real constant and $\alpha$ be an arbitrary solution to the KG equation of vortex filament (isolated-stretched, self-induction, elasticity) in an ideal fluid. The most general solution (3.1) gives us all possible most general infinitesimal symmetries of KG equation.
References


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