New Fixed Point Theorems for Generalized Distances

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Abstract

In the recent developments of fixed point theorems is proving the existence of fixed points on partially ordered metric spaces [20, 23], a generalization of the Banach contraction principle for an integral-type inequality [2, 10, 21], and even some applications to matrix equations and ordinary differential equations. In this paper, we want to illustrate some new fixed point theorems for generalized distances with $\tau$-function in the partially ordered metric space rather than $w$-distance [23]. In short, we would like to utilize $\tau$-function to provide the existence of fixed points on partially ordered metric spaces which is our goal.

Keywords: $w$-distance, $\tau$-function, fixed point theorem

1. Introduction and preliminaries

One of the amazing features of twentieth century mathematics has been its recognition of the power of the abstract approach. The term "abstract" is a highly subjective one. This has given rise to a large body of new results and problems and has, in fact, led us to open up whole new areas of mathematics whose very existence had not even been suspected.

In the wake of these developments has come not only a new mathematics but a fresh outlook, and along with this, simple new proofs of difficult classical
result. The isolation of a problem into its basic essentials has often revealed for us the proper setting, in the whole scheme of things, of results considered to have been special and apart and has shown us interrelations between areas previously thought to have been unconnected.

In 1922, the Polish mathematician Stefan Banach had illustrated the famous theorem: contraction mapping principle (see, e.g. [26]), which sometimes also known as the Banach fixed point theorem. This beautiful and powerful result guaranteed the existence and uniqueness of fixed points, and provided fundamental building blocks to find these fixed points by utilizing the constructive method which will provoke the sense of aesthetics in mathematics! Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done. In 1969, Sam B. Nadler, Jr. [18] had proven a set-valued generalized vision of the Banach fixed point theorem. After that, in 1989, Mizoguchi and Takahashi [17] had proven a generalization of Nadler’s fixed point theorem, and soon we will give a quick review as following. However, before reading this result, we should first take a quick look for some materials ahead.

Let \((X, d)\) be a metric space and \(p : X \times X \to [0, \infty)\) be any function. For each \(x \in X\) and \(A \subseteq X\), let

\[
d(x, A) = \inf_{y \in A} d(x, y)
\]

and

\[
p(x, A) = \inf_{y \in A} p(x, y).
\]

Denote by \(N(X)\) the class of all nonempty subsets of \(X\), \(C(X)\) the family of all nonempty closed subsets of \(X\) and \(CB(X)\) the family of all nonempty closed and bounded subsets of \(X\). A function \(H : CB(X) \times CB(X) \to [0, \infty)\) defined by

\[
H(A, B) = \max\left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},
\]

is said to be the Hausdorff metric on \(CB(X)\) induced by the metric \(d\) on \(X\). A point \(v\) in \(X\) is a fixed point of a map \(T : X \to X\) if \(Tv = v\).

Now we turn around and come back to see the famous Nadler’s fixed point theorem.

**Theorem 1.1.** [18] (Nadler)
Let $(X, d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $\lambda \in [0, 1)$. Assume that
\[ \mathcal{H}(Tx, Ty) \leq \lambda d(x, y) \]
for all $x, y \in X$. Then $\mathcal{F}(T) \neq \emptyset$.

A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $MT$-function (or $R$-function) [5-8] if
\[ \limsup_{s \to t^+} \varphi(s) < 1 \]
for all $t \in [0, \infty)$.

The following is a well-known generalization of Nadler’s fixed point theorem.

**Theorem 1.2.** [17] (Mizoguchi and Takahashi)

Let $(X, d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $\varphi : [0, \infty) \to [0, 1)$ be a $MT$-function. Assume that
\[ \mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) \]
for all $x, y \in X$. Then $\mathcal{F}(T) \neq \emptyset$.

Until to 2007, M. Berinde and V. Berinde [1] gave a generalization of Mizoguchi-Takahashi’s fixed point theorem.

**Theorem 1.3.** [1] (M. Berinde and V. Berinde)

Let $(X, d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued map, $\varphi : [0, \infty) \to [0, 1)$ be a $MT$-function and $L \geq 0$. Assume that
\[ \mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx) \]
for all $x, y \in X$. Then $\mathcal{F}(T) \neq \emptyset$.

The concept of $w$-distances was introduced by Kada, Suzuki and Takahashi [11, 15, 26]. Here we give a brief review. A function $p : X \times X \to [0, \infty)$ is called a $w$-distance, if the following are satisfied:

1. $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
2. for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is l.s.c.;
3. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$. 
The $w$-distances was primarily utilized to prove Caristi’s fixed point theorem [3], Ekeland’s variational principle (see, e.g. [26]).

A function $p : X \times X \rightarrow [0, \infty)$ is said to be a $\tau$-т-unction [4, 5, 7, 8, 14, 16], introduced and studied by Lin and Du, if the following conditions hold:

$(\tau_1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

$(\tau_2)$ if $x \in X$ and $\{y_n\}$ in $X$ with $\lim_{n \rightarrow \infty} y_n = y$ such that $p(x, y_n) \leq M$ for some $M = M(x) > 0$, then $p(x, y) \leq M$;

$(\tau_3)$ for any sequence $\{x_n\}$ in $X$ with $\lim_{n \rightarrow \infty} \sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in $X$ such that $\lim_{n \rightarrow \infty} p(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$;

$(\tau_4)$ for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Let $(X, d)$ be a metric space. A function $p : X \times X \rightarrow [0, \infty)$ is called a $\tau^0$-т-unction [4, 5, 7, 8] if it is a $\tau$-t-unction on $X$ with $p(x, x) = 0$ for all $x \in X$.

**Definition 1.1.**

Let $X$ be a nonempty set. Then $(X, d, \preceq)$ is called a partially ordered metric space if

$(\rho_1)$ $(X, \preceq)$ is a partially ordered set,

$(\rho_2)$ $(X, d)$ is a metric space.

**Definition 1.2.**

Let $(X, \preceq)$ be a partially ordered set, $T : X \rightarrow X$ be a function and $x, y \in X$. Then we call

$(1)$ $x$ and $y$ are comparable with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$.

$(2)$ $T$ is nondecreasing with respect to $\preceq$ if $x \preceq y$ implies $Tx \preceq Ty$.

Let $(X, \preceq)$ be a partially ordered set. We give the following notation:

$(i)$ $X_{\preceq} = \{(x, y) \in X \times X|x \preceq y$ or $y \preceq x\}$,

$(ii)$ $\Phi = \{\phi|\phi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, continuous and $\phi(t) > 0$ for all $t > 0\}$,
(iii) $\Psi = \{ \psi : (0, \infty) \rightarrow (0, \infty) \text{ is nondecreasing, right continuous and } \psi(x) < x \text{ for all } x > 0 \}$. 

In the recent developments of fixed point theorems is proving the existence of fixed points on partially ordered metric spaces [20, 23], a generalization of the Banach contraction principle for an integral-type inequality [2, 10, 21], and even some applications to matrix equations and ordinary differential equations, and others, see [9, 12, 13, 19, 20, 22, 24, 25, 27] In this paper, we want to illustrate some new fixed point theorems for generalized distances with $\tau^0$-function in the partially ordered metric space. At this point, we introduce some crucial definitions and lemmas.

Lemma 1.1. [23]
If $\psi \in \Psi$, then
$$\lim_{n \to \infty} \psi^n(\epsilon) = 0.$$  

Lemma 1.2. [23]
If $\phi \in \Phi$, $\{a_n\} \subset [0, \infty)$ and
$$\lim_{n \to \infty} \phi(a_n) = 0,$$  
then $\lim_{n \to \infty} a_n = 0$.

Lemma 1.3. [4, 5, 7, 8, 14, 16]
Let $(X, d)$ be a metric space and $p : X \times X \to [0, \infty)$ be a function. Assume that $p$ satisfies the condition $(\tau 3)$. If a sequence $\{x_n\}$ in $X$ with $\lim_{n \to \infty} \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in $X$.

Lemma 1.4. [5, 8]
Let $A$ be a closed subset of a metric space $(X, d)$ and $p : X \times X \to [0, \infty)$ be any function. Suppose that $p$ satisfies $(\tau 3)$ and there exists $u \in X$ such that $p(u, u) = 0$. Then $p(u, A) = 0$ if and only if $u \in A$.

2. Main results

In this section, we will establish some new fixed point theorems for $\tau^0$-functions.

Theorem 2.1.
Let $(X, d, \preceq)$ be a complete partially ordered metric space, $p$ be a $\tau^0$-function and $T : X \to X$ be a nondecreasing mapping. Suppose that
(α) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\leq$,

(β) there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that
\[
\phi(p(Tx, Ty)) \leq \psi\phi(p(x, y))
\]
for all $(x, y) \in X_\leq$,

(γ) the mapping $f : X \to [0, \infty)$ defined by
\[
f(x) = p(x, Tx)
\]
is lower semi-continuous.

Then $F(T) \neq \emptyset$.

**Proof.**

By the condition (α), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\leq$. If $x_0 = Tx_0$, then $F(T) \neq \emptyset$ and therefore the theorem is finished. Otherwise, we have to consider the other case $x_0 \neq Tx_0$. Define $x_n = T^n x_0$ for $n \in \mathbb{N}$. Since $p$ is a $\tau^0$-function, $p(x_0, x_1) > 0$. According to the monotonicity of $T$, we have $(x_1, x_2) \in X_\leq$. If we utilize this processing inductively, then we could attain

\[
(x_n, x_m) \in X_\leq
\]
for any $n, m \in \mathbb{N}$. Now, for any $n \in \mathbb{N}$, we apply the condition (β) to achieve
\[
\phi(p(x_n, x_{n+1})) \leq \psi\phi(p(x_{n-1}, x_n)) \leq \psi^2\phi(p(x_{n-2}, x_{n-1})) \leq \ldots \leq \psi^n\phi(p(x_0, x_1)).
\]

Now we employ Lemma 1.1 to obtain
\[
\lim_{n \to \infty} \psi^n(\phi(p(x_0, x_1))) = 0.
\]

Taking limit as $n \to \infty$ in the last inequality, we have
\[
\lim_{n \to \infty} \phi(p(x_n, x_{n+1})) = 0.
\]

By Lemma 1.2 to attain
\[
\lim_{n \to \infty} p(x_n, x_{n+1}) = 0. \quad (2)
\]

Similarly, we could utilize the same arguments to get
\[
\lim_{n \to \infty} p(x_{n+1}, x_n) = 0. \quad (3)
\]
In the next stage we would like to claim
\[ \lim_{n,m \to \infty} p(x_n, x_m) = 0. \] (4)

Assume that our claim is not true. Then we could find \( \delta > 0 \) and sequences \( \{m(k)\}, \{n(k)\} \) such that for all positive integers \( k \),
\[ n(k) > m(k) > k, \]
\[ p(x_{n(k)}, x_{m(k)}) \geq \delta \] (5)
and
\[ p(x_{n(k)}, x_{m(k)-1}) < \delta. \] (6)

From (2) there exists \( k_0 \in \mathbb{N} \) such that \( k > k_0 \) implies
\[ p(x_{n(k)}, x_{n(k)+1}) < \delta. \] (7)

In the view of (5) and (7), we have \( m(k) \neq n(k) + 1 \) for all \( k > k_0 \). Thanks to (5) and (6), we get
\[ 0 < \delta \leq p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) < \delta + p(x_{m(k)-1}, x_{m(k)}), \]
for \( k \in \mathbb{N} \). Therefore, by (2) and the last inequality, we conclude that
\[ \lim_{k \to \infty} p(x_{n(k)}, x_{m(k)}) = \delta. \] (8)

For each \( k \in \mathbb{N} \), since
\[ p(x_{n(k)}, x_{m(k)}) \leq p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{m(k)+1}) + p(x_{m(k)+1}, x_{m(k)}) \]
and
\[ p(x_{n(k)+1}, x_{m(k)+1}) \leq p(x_{n(k)+1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)}, x_{m(k)+1}), \]
by (2), (3) and (8), we have
\[ \limsup_{k \to \infty} p(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k \to \infty} p(x_{n(k)}, x_{m(k)}) = \delta. \]
Let $\lambda_k = p(x_{n(k)+1}, x_{m(k)+1})$, $k \in \mathbb{N}$. Then
\[
\limsup_{k \to \infty} \lambda_k = \delta.
\]
Hence there exists a subsequence $\{\lambda_{k(r)}\} \subset \{\lambda_k\}$ such that
\[
\lim_{r \to \infty} \lambda_{k(r)} = \delta.
\]
That is,
\[
\lim_{r \to \infty} p(x_{n(k(r))+1}, x_{m(k(r))+1}) = \delta. \tag{9}
\]
Since $\phi$ is continuous, by (8) and (9), we could obtain
\[
\phi(\delta) = \lim_{r \to \infty} \phi(p(x_{n(k(r))+1}, x_{m(k(r))+1})), \tag{10}
\]
and when $r \to \infty$,
\[
\phi(p(x_{n(k(r)}, x_{m(k(r)})) \downarrow \phi(\delta). \tag{11}
\]
Since $(x_{n(k(r)}), x_{m(k(r)}) \in X$ and $\psi$ is right continuous, from (10), (11) and condition $(\beta)$, we have
\[
\phi(\delta) = \lim_{r \to \infty} \phi(p(x_{n(k(r)+1}, x_{m(k(r)+1)})) \leq \psi(\phi(\delta)),
\]
which implies $\phi(\delta) = 0$. However, by the definition of $\phi$, we have
\[
0 = \phi(\delta) > 0,
\]
a contradiction. Therefore, we have finished our claim (4)
\[
\lim_{n,m \to \infty} p(x_n, x_m) = 0.
\]
From (4), we get
\[
\limsup_{n \to \infty} \{p(x_n, x_m) : m > n\} = 0.
\]
By Lemma 1.3, $\{x_n\}$ is a Cauchy sequence in $X$. It is essential to point out that $X$ is a complete metric space, the key point here is there must exist $v \in X$ such that $x_n \to v$ as $n \to \infty$.

In the final stage, we apply straightforwardly condition $(\gamma)$ to this placement. Since we have already required the lower semicontinuity of $f$ and $x_n \to v$ as $n \to \infty$, we could attain
\[
p(v, Tv) = f(v) \leq \liminf_{n \to \infty} f(x_n) = \liminf_{n \to \infty} p(x_n, Tx_n) \leq \lim_{n \to \infty} p(x_n, x_{n+1}) = 0,
\]
which implies $p(v, Tv) = 0$. From Lemma 1.4, we have achieved that $v \in \mathcal{F}(T)$. Therefore we have completed our proof. \hfill \square

**Theorem 2.2.**

Let $(X, d, \preceq)$ be a complete partially ordered metric space, $p$ be a $\tau^0$-function and $T : X \to X$ be a nondecreasing mapping. Suppose that

(I) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\preceq$,

(II) there exists $\psi \in \Psi$ such that

$$p(Tx, Ty) \leq \psi(p(x, y))$$

(12)

for all $(x, y) \in X_\preceq$,

(III) the mapping $f : X \to [0, \infty)$ defined by

$f(x) = p(x, Tx)$

is lower semi-continuous.

Then $\mathcal{F}(T) \neq \emptyset$.

**Proof.**

Let $\phi : [0, \infty) \to [0, \infty)$ be defined by

$$\phi(t) = t.$$

Then $\phi \in \Phi$. Hence (12) implies

$$\phi(p(Tx, Ty)) \leq \psi(p(x, y))$$

for all $(x, y) \in X_\preceq$. So all assumptions of Theorem 2.1 are satisfied. Therefore the conclusion follows from Theorem 2.1. \hfill \square

Since any metric $d$ is a $\tau^0$-function, we could obtain immediately the following results.

**Theorem 2.3.**

Let $(X, d, \preceq)$ be a complete partially ordered metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\preceq$,
(ii) there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that
\[
\phi(d(Tx, Ty)) \leq \psi \phi(d(x, y))
\]
for all $(x, y) \in X_\preceq$.

(iii) the mapping $f : X \rightarrow [0, \infty)$ defined by
\[
f(x) = d(x, Tx)
\]
is lower semi-continuous.

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Theorem 2.4.

Let $(X, d, \preceq)$ be a complete partially ordered metric space and $T : X \rightarrow X$ be a nondecreasing mapping. Suppose that

(1) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in X_\preceq$,

(2) there exists $\psi \in \Psi$ such that
\[
d(Tx, Ty) \leq \psi(d(x, y)) \tag{13}
\]
for all $(x, y) \in X_\preceq$,

(3) the mapping $f : X \rightarrow [0, \infty)$ defined by
\[
f(x) = d(x, Tx)
\]
is lower semi-continuous.

Then $\mathcal{F}(T) \neq \emptyset$.

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