On the Quadratic Additive Type Functional Equations

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Abstract

In this paper, we investigate the quadratic additive type functional equations.

Mathematics Subject Classification: 39B82, 39B52

Keywords: quadratic additive type functional equations, quadratic-additive mapping

1 Introduction

Throughout this paper, let $G$ be an abelian group and $V$ a real vector space. For a given mapping $f : G \to V$, we define

$$Af(x, y) := f(x + y) - f(x) - f(y),$$

$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)$$

for all $x, y \in G$. A mapping $f : G \to V$ is called an additive mapping (a quadratic mapping, respectively) if $f$ satisfies the functional equation $Af = 0$ ($Qf = 0$, respectively). Observe that the mappings $g, h : R \to R$ given by $g(x) = ax$ and $h(x) = ax^2$ are solutions of $Ag(x, y) = 0$ and $Qh(x, y) = 0$, respectively. On the other hand if a mapping is represented by the sum of an
additive mapping and a quadratic mapping, we call the mapping a quadratic-additive mapping. For a functional equation $Ef = 0$ if all of the solutions of $Ef = 0$ are quadratic-additive mappings and all of quadratic-additive mappings are the solutions of $Ef = 0$, then we call the functional equation $Ef = 0$ a quadratic additive type functional equation. The mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx$ is a solution of the quadratic additive type functional equation. The stability problems of quadratic additive type functional equations have been extensively investigated by a number of mathematicians, see [1]-[32].

In this paper, I introduce many kinds of the quadratic additive type functional equations and prove that there is a unique quadratic-additive mapping under the some conditions.

2 Quadratic-additive type functional equations

For a given mapping $f : G \to V$, we use the following abbreviations:

$E_1 f(x,y,z,w) = f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w)
- f(x+y) - f(x+z) - f(x+w)
- f(y+z) - f(y+w) - f(z+w)$,

$E_2 f(x,y,z) = f(x+y+z) - f(x+y) - f(y+z) - f(x+z)
+ f(x) + f(y) + f(z)$,

$E_3 f(x_1, \cdots, x_n) = f(\sum_{j=1}^{n} x_j) + (n-2)\sum_{j=1}^{n} f(x_j)
- \sum_{1 \leq i < j \leq n} f(x_i + x_j)$,

$E_4 f(x,y) = 2f(x+y) + f(x-y) + f(y-x)
- 3f(x) - f(-x) - 3f(y) - f(-y)$,

$E_5 f(x,y) = \sum_{\delta_2=0}^{1} \cdots \sum_{\delta_n=0}^{1} f\left(x_1 + \sum_{j=2}^{n} (-1)^{\delta_j} x_j\right)
- 2^{n-1} f(x_1) - 2^{n-2} \sum_{j=2}^{n} (f(x_j) + f(-x_j))$,

$E_6 f(x,y) = 2f(x+y) + f(x-y) + f(y-x) - f(2x) - f(2y)$,

$E_7 f(x_1, \cdots, x_n) = 2f(\sum_{j=1}^{n} x_j)
+ \sum_{1 \leq i \leq n, i \neq j} f(x_i - x_j)
- (n+1)\sum_{j=1}^{n} f(x_j)
- (n-1)\sum_{j=1}^{n} f(-x_j)$,

$E_8 f(x,y) = f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)$,

$E_9 f(x,y) = 3f(x+2y) + 3f(x-2y) - 6f(x) - 4f(2y) - 8f(-y)$,

$E_{10} f(x,y) = f(x+y) - f(x-y) + f(x-2y)$
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\[ E_{11} f(x, y) = f(x + 2y) - f(-x - y) - f(x + y) + f(-x) - 2f(y), \]
\[ E_{12} f(x, y, z, w) = f(x + y + z + w) + f(x - y - z + w) + f(x + y - z - w) \]
\[ + f(x + y + z - w) + f(x + y - z + w) + f(x - y + z - w) \]
\[ + f(x + y - z - w) + f(x - y - z - w) - 8f(x) \]
\[ - 4f(y) - 4f(-y) - 4f(z) - 4f(-z) - 4f(w) - 4f(-w), \]
\[ E_{13} f(x, y, z, w) = f(-x - y + z + w) + f(x - y + z + w) + f(x + y - z + w) \]
\[ + f(-x + y + z - w) + f(-x + y - z + w) + f(x - y + z - w) \]
\[ + 2f(x + y - z + w) - 5f(x) - 3f(-x) - 5f(y) \]
\[ - 3f(-y) - 5f(z) - 3f(-z) - 5f(w) - 3f(-w), \]
\[ E_{14} f(x_1, \cdots, x_n) = \sum_{1 \leq i, j \leq n, i \neq j} [f(x_i + x_j) + f(x_i - x_j)] \]
\[ - (n - 1) \sum_{j=1}^{n} [3f(x_j) + f(-x_j)], \]
\[ E_{15} f(x_1, \cdots, x_n) = \sum_{1 \leq i, j \leq n, i \neq j} [f(x_i + x_j) + f(x_i - x_j)] \]
\[ - (n - 1) \sum_{j=1}^{n} f(2x_j), \]
\[ E_{16} f(x, y, z) = f(x + y + z) + f(x - y + z) + f(x + y - z) \]
\[ + f(-x + y + z) - 3f(x) - f(-x) \]
\[ - 3f(y) - f(-y) - 3f(z) - f(-z), \]
\[ E_{17} f(x, y, z) = f(x + y + z) + f(x - y - z) + f(x - y + z) \]
\[ + f(x + y - z) - 4f(x) - 2f(y) \]
\[ - 2f(-y) - 2f(z) - 2f(-z), \]
\[ E_{18} f(x, y, z, w) = f(-x + y + z + w) + f(x - y + z + w) + f(x + y - z + w) \]
\[ + f(x + y + z - w) - 3f(x) - f(-x) - 3f(y) \]
\[ - f(-y) - 3f(z) - f(-z) - 3f(w) - f(-w), \]
\[ E_{19} f(x, y) = \frac{nC_k}{n - 1} f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f \left( \sum_{i=1}^{n} x_i - 2 \sum_{j=1}^{k} x_{i_j} \right) \]
\[ - nC_k \sum_{i=1}^{n} \left( \frac{n + 1}{2(n - 1)} f(x_i) + \frac{1}{2} f(-x_i) \right) \quad (n = 2k), \]
\[ E_{20} f(x, y) = \frac{nC_k}{n} f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} f \left( \sum_{i=1}^{n} x_i - 2 \sum_{j=1}^{k} x_{i_j} \right) \]
\[ - nC_k \sum_{i=1}^{n} \left( \frac{k + 2}{n} f(x_i) + \frac{k}{n} f(-x_i) \right), \quad (n = 2k + 1) \]
for all \( x, y, z, w, x_1, x_2, \cdots, x_n \in G \).

We need the following lemmas to prove Theorem 2.3 and Theorem 2.6.

**Lemma 2.1** If a mapping \( f : G \to V \) satisfies one of the functional equations \( E_i f = 0, i = 1, 2, \cdots, 20 \), then \( f \) is an additive mapping, where \( f_o(x) := \frac{f(x) - f(-x)}{2} \) for all \( x \in G \).

**Proof.** The result follows from the following equalities

\[
Af_o(x, y) = -\frac{1}{2} E_1 f_o(x, y, x, y) - \frac{1}{4} E_1 f_o(x + y, x + y, x + y, -x - y) + \frac{1}{4} E_1 f_o(x, x, -x) + \frac{1}{4} E_1 f_o(y, y, y, -y),
\]

\[
Af_o(x, y) = -\frac{1}{2} E_2 f_o(x, y, -x - y),
\]

\[
Af_o(x, y) = \frac{1}{2} E_3 f_o(x + y, -x, -y, 0, \cdots, 0),
\]

\[
Af_o(x, y) = \frac{1}{2} E_4 f_o(x, y),
\]

\[
Af_o(x, y) = \frac{1}{2n} \left( E_5 f_o(x, x, 0, \cdots, 0) + E_5 f_o(y, y, 0, \cdots, 0) - E_5 f_o(x + y, x - y, 0, \cdots, 0) \right),
\]

\[
Af_o(x, y) = \frac{1}{2} \left( E_6 f_o(x, y) - E_6 f_o(x, 0) - E_6 f_o(0, y) \right),
\]

\[
Af_o(x, y) = \frac{1}{2} E_7 f_o(x, y, 0, \cdots, 0),
\]

\[
Af_o(x, y) = -E_8 f_o(x + y, x - y) + E_8 f_o(x, x) + E_8 f_o(y, y),
\]

\[
Af_o(x, y) = -\frac{1}{12} E_9 f_o(2x + 2y, x - y) + \frac{1}{12} E_9 f_o(0, x - y) + \frac{1}{8} E_9 f_o(0, x + y) - \frac{1}{16} E_9 f_o(0, 2x) - \frac{1}{8} E_9 f_o(0, x) - \frac{1}{16} E_9 f_o(0, 2y) - \frac{1}{8} E_9 f_o(0, y),
\]

\[
Af_o(x, y) = -\frac{1}{4} E_{10} f_o(2x + 2y, x - y) - \frac{1}{4} E_{10} f_o(2x + 2y, y - x) + \frac{1}{2} E_{10} f_o(0, x + y) - \frac{1}{4} E_{10} f_o(0, 2x) - \frac{1}{2} E_{10} f_o(0, x) - \frac{1}{4} E_{10} f_o(0, 2y) - \frac{1}{2} E_{10} f_o(0, y),
\]

\[
Af_o(x) = \frac{1}{2} \left( E_{11} f_o(2x, y) - E_{11} f_o(0, x + y) + E_{11} f_o(0, x) + E_{11} f_o(0, y) \right),
\]

\[
Af_o(x, y, z, w) = \frac{E_{12} f_o(x, x, 0, 0) + E_{12} f_o(y, y, 0, 0) - E_{12} f_o(x + y, y, -y, 0, 0)}{8},
\]
\[ Af_0(x, y, z, w) = \frac{E_{13}f_0(x, y, 0, 0)}{2}, \]
\[ Af_o(x, y) = \frac{1}{2}E_{14}f_0(x, y, 0, 0), \]
\[ Af_0(x, y) = \frac{1}{2}E_{15}f_0(x, y, 0, 0) - \frac{1}{2}E_{15}f_0(x, 0, 0, 0) \]
\[ - \frac{1}{2}E_{15}f_0(0, y, 0, 0, 0), \]
\[ Af_0(x, y, z) = \frac{E_{16}f_0(x, y, 0)}{2}, \]
\[ Af_0(x, y, z) = \frac{1}{4}E_{17}f_0(x, x, 0) + \frac{1}{4}E_{17}f_0(y, y, 0) - \frac{1}{4}E_{17}f_0(x + y, x - y, 0), \]
\[ Af_0(x, y, z, w) = \frac{1}{2}E_{18}f_0(x, y, 0, 0), \]
\[ Af_0(x, y) = \frac{n - 1}{nC_k}E_{19}f_0(x, y, 0, 0, 0), \]
\[ Af_0(x, y) = \frac{n}{2 \cdot nC_k}E_{20}f_0(x, y, 0, 0, 0) \]

for all \( x, y \in G \).

**Lemma 2.2** If a mapping \( f : G \rightarrow V \) satisfies one of the functional equations \( E_if = 0, \ i = 1, 2, \ldots, 20 \), then \( f_e \) is a quadratic mapping, where \( f_e(x) := \frac{f(x) + f(-x)}{2} \) for all \( x \in G \).

**Proof.** The result follows from the following equalities
\[ Qf_e(x, y) = -\frac{1}{2}E_1f_e(x, y, -x, -y) - \frac{1}{6}E_1f_e(0, 0, 0, 0), \]
\[ Qf_e(x, y) = -E_2f_e(x, y, -x) - E_2f_e(0, 0, 0, 0), \]
\[ Qf_e(x, y) = -E_3f_e(x, y, -y, 0, \ldots, 0) - \frac{n^2 - 3n - 2}{n^2 - 3n + 2}E_3f_e(0, 0, \ldots, 0), \]
\[ Qf_e(x, y) = \frac{1}{2}E_4f_e(x, y), \]
\[ Qf_e(x, y) = E_5f_e(x, y, 0, \ldots, 0) - \frac{n^2}{n - 2}E_5f_e(0, 0, \ldots, 0), \]
\[ Qf_e(x, y) = \frac{1}{2}E_6f_e(x, y, 0, \ldots, 0) - E_6f_e(0, y) - E_6f_e(x, 0, 0), \]
\[ Qf_e(x, y) = \frac{1}{2}E_7f_e(x, y, 0, \ldots, 0) - \frac{(n - 2)(n + 3)}{n^2 + n - 2}E_7f_e(0, 0, \ldots, 0), \]
\[ Qf_e(x, y) = E_8f_e(x, y), \]
\[ Qf_e(x, y) = \frac{1}{12}E_9f_e(2x, y) - \frac{1}{8}E_9f_e(0, x + y) - \frac{1}{8}E_9f_e(0, x - y) \]
\[ + \frac{1}{4}E_9f_e(0, x) + \frac{1}{6}E_9f_e(0, y) - \frac{1}{12}E_9f_e(0, 0). \]
\[ Qf_e(x, y) = \frac{1}{4} E_{10} f_e(2x, y) + \frac{1}{4} E_{10} f_e(2x, -y) - \frac{1}{4} E_{10} f_e(0, x - y) \]
\[ - \frac{1}{4} E_{10} f_e(0, x + y) + \frac{1}{2} E_{10} f_e(0, x), \]
\[ Qf_e(x, y) = E_{11} f_e(x - y, y), \]
\[ Qf_e(x, y, z, w) = \frac{3E_{12} f_e(x, y, 0, 0) - 2E_{12} f_e(0, 0, 0, 0)}{12}, \]
\[ Qf_e(x, y, z, w) = \frac{3E_{13} f_e(x, y, 0, 0) - 2E_{13} f_e(0, 0, 0, 0)}{3}, \]
\[ Qf_e(x, y) = \frac{1}{2} E_{14} f_e(x, y, 0, \ldots, 0) - \frac{n - 2}{2n} E_{14} f_e(0, 0, \ldots, 0), \]
\[ Qf_e(x, y) = \frac{1}{2} E_{15} f_e(x, y, 0, \ldots, 0) - \frac{1}{2} E_{15} f_e(0, y, 0, \ldots, 0) \]
\[ - \frac{1}{2} E_{15} f_e(x, 0, 0, \ldots, 0) - \frac{n^2 - n - 4}{2(n^2 - n)} E_{15} f_e(0, 0, \ldots, 0), \]
\[ Qf_e(x, y, z) = \frac{2E_{16} f_e(x, y, 0) - E_{16} f_e(0, 0, 0)}{4}, \]
\[ Qf_e(x, y, z) = \frac{2E_{17} f_e(x, y, 0) - E_{17} f_e(0, 0, 0)}{4}, \]
\[ Qf_e(x, y, z, w) = \frac{3E_{18} f_e(x, y, 0, 0) - 2E_{18} f_e(0, 0, 0, 0)}{6}, \]
\[ Qf_e(x, y) = \frac{n - 1}{k \cdot nC_k} E_{19} f_e(x, y, 0, \ldots, 0) \]
\[ - \frac{n - 2}{k \cdot nC_k} E_{19} f_e(0, 0, 0, \ldots, 0), \]
\[ Qf_e(x, y) = \frac{1}{nC_k} E_{20} f_e(x, y, 0, \ldots, 0) \]
\[ - \frac{n - 2}{nC_k(n - 1)} E_{20} f_e(0, 0, 0, \ldots, 0) \]

for all \( x, y \in G \).

**Theorem 2.3** If a mapping \( f : G \to V \) satisfies one of the functional equations \( E_i f = 0, i = 1, \ldots, 20 \), then \( f \) is a quadratic-additive mapping.

**Proof.** Since \( f = f_e + f_o \), the results follows from Lemma 2.1 and Lemma 2.2.

**Lemma 2.4** If \( f : G \to V \) is a quadratic mapping, then \( f \) satisfies the functional equations \( E_i f = 0, i = 1, 2, 3, \ldots, 18 \).

**Proof.** Assume that \( f \) is quadratic mapping. From the following equalities

\[ E_1 f(x, y, z, w) = \frac{1}{2}(Qf(x + y, z + w) + Qf(x + w, y + z) \]

...
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\[ E_2 f(x, y, z) = \frac{1}{2} (Qf(x + y, z) - Qf(x - z, y) + Qf(x, y + z) - Qf(x, z)), \]

\[ E_4 f(x, y) = Qf(x, y) + \frac{1}{2} (Qf(x, -y) + Qf(y, -x)), \]

\[ E_5 f(x_1, \ldots, x_n) = \sum_{\delta_3=0}^1 \cdots \sum_{\delta_n=0}^1 Qf \left( x_1, x_2 + \sum_{j=3}^n (-1)^{\delta_j} x_j \right) \]
\[ + 2 \sum_{\delta_1=0}^1 \sum_{\delta_n=0}^1 Qf \left( x_2, x_3 + \sum_{j=4}^n (-1)^{\delta_j} x_j \right) \]
\[ + \cdots \]
\[ + 2^{n-3} \sum_{\delta_n=0}^1 Qf \left( x_{n-2}, x_{n-1} + \sum_{j=n}^n (-1)^{\delta_j} x_j \right), \]
\[ + 2^{n-2} Qf(x_{n-1}, x_n) + 2^{n-2} \sum_{j=2}^n Qf(0, x_j) \]
\[ - 2^{n-2} nQf(0, 0), \]

\[ E_6 f(x, y) = -\frac{1}{2} (Qf(x + y, x - y) + (Qf(x + y, y - x)), \]

\[ E_8 f(x, y) = \frac{1}{2} (Qf(x, y) + Qf(x, -y)), \]

\[ E_9 f(x, y) = 3Qf(x, 2y) + 2Qf(y, -y) - 4Qf(0, y) + 5Qf(0, 0), \]

\[ E_{10} f(x, y) = Qf(x - y, y) + Qf(x, -y), \]

\[ E_{11} f(x, y) = Qf(x + y, y) - Qf(0, x + y) + Qf(0, x), \]

\[ E_{12} f(x, y, z, w) = Qf(x + y, z - w) + Qf(x - y, z + w) + Qf(x - y, z - w) \]
\[ + Qf(x + y, z + w) + 4Qf(x, y) + 4Qf(z, w) \]
\[ - 4Qf(0, y) - 4Qf(0, z) - 4Qf(0, w) + 12Qf(0, 0), \]

\[ E_{13} f(x, y, z, w) = Qf(x + y, z + w) + Qf(x - y, z - w) + Qf(z + w, x + y) \]
\[ + Qf(z - w, x - y) + \frac{5}{2} Qf(x, y) + \frac{3}{2} Qf(x, y) - \frac{3}{2} Qf(0, x) \]
\[ + \frac{5}{2} Qf(z, w) + \frac{3}{2} Qf(z, w) - \frac{3}{2} Qf(0, z) + 3Qf(0, 0), \]

\[ E_{14} f(x_1, \ldots, x_n) = \frac{1}{2} \sum_{1 \leq i, j \leq n, i \neq j} (Qk(x_i, x_j) + Qk(x_i, -x_j)), \]

\[ E_{15} f(x_1, \ldots, x_n) = \frac{1}{2} \sum_{1 \leq i, j \leq n, i \neq j} (Qk(x_i, x_j) + Qk(x_i, -x_j)) \]
\[ - \frac{n-1}{2} \sum_{i=1}^n (Qk(x_i, x_i) + Qk(x_i, -x_i)), \]

\[ E_{16} f(x, y, z) = Qf(x + y, z) + Qf(z, x - y) + 2Qf(x, y) \]
for all \(x, y, z, w, x_1, x_2, \ldots, x_n \in G\), we can show that \(f\) satisfies the functional equations \(E_i f = 0\), where \(i = 1, 2, 4, 5, 6, 8, \ldots, 18\).

Now we want to prove \(E_3 f = 0\) and \(E_7 f = 0\) to complete the proof. We apply an induction on \(j \in \{2, 3, \ldots, n\}\) to prove \(E_3 f(x_1, x_2, \ldots, x_n) = 0\) and \(E_7 f(x_1, x_2, \ldots, x_n) = 0\) for all \(x_1, x_2, \ldots, x_n \in G\). For \(j = 2\), we have

\[
E_3 f(x_1, x_2, 0, \ldots, 0) = \frac{n-2}{n-1} Qf(0, 0) = 0,
\]

\[
E_7 f(x_1, x_2, 0, \ldots, 0) = 2Qf(x_1, x_2) - Qf(0, 0) = 0.
\]

We may assume that \(n > 2\). If \(E_3 f(x_1, \ldots, x_j, 0, \ldots, 0) = 0\) and \(E_7 f(x_1, \ldots, x_j, 0, \ldots, 0) = 0\) for some integer \(j\) \((2 \leq j < n)\) and for all \(x_1, \ldots, x_j \in G\), then a routine calculation yields

\[
E_3 f(x_1, x_2, \ldots, x_{j+1}, 0, \ldots, 0) = Qf(x_1 + \cdots + x_{j+1}, -x_2 - \cdots - x_j + x_{j+1})
\]

\[
- E_3 f(x_1, 2x_2, \ldots, 2x_j, 0, \ldots, 0) + 2E_3 f(-x_2, \ldots, -x_j, x_{j+1} 0, \ldots, 0)
\]

\[
- \sum_{i=1}^{j+1} Qf(x_1 + x_i, x_i) - \sum_{i=2}^{j-1} 2Qf(x_i, x_{j+1})
\]

\[
- \sum_{1 < i < k < j+1} Q(f(x_i + x_k, x_i + x_k) - (j-2)Qf(x_2, x_2) - (j-2)Qf(x_j, x_j)
\]

\[
= 0,
\]

\[
E_7 f(x_1, x_2, \ldots, x_{j+1}, 0, \ldots, 0) = Qf(x_1 + \cdots + x_j, x_{j+1} - x_j) + \frac{1}{2} E_7 f(x_1, x_2, \ldots, x_{j-1}, 2x_j, 0, \ldots, 0)
\]

\[
+ \frac{1}{2} E_7 f(x_1, x_2, \ldots, x_{j-1}, 2x_{j+1}, 0, \ldots, 0)
\]

\[
- \sum_{i=1}^{j-1} (Qf(x_i, x_j + Qf(x_i, x_{j+1}) - \frac{j}{2} Qf(x_{j+1}, x_{j+1}) - \frac{j}{2} Qf(x_j, x_j)
\]

\[
= 0
\]

for all \(x_1, x_2, \ldots, x_{j+1} \in G\). Hence, we conclude that

\[
E_3 f(x_1, x_2, \ldots, x_n) = 0, \quad E_7 f(x_1, x_2, \ldots, x_n) = 0
\]

for all \(x_1, x_2, \ldots, x_n \in G\).
Lemma 2.5 If $f : G \rightarrow V$ is an additive mapping, then $f$ satisfies the functional equations $E_i f = 0$, where $i = 1, \cdots, 20$.

Proof. If $f$ is an additive mapping, then we can easily show that

$$f \left( \sum_{j=1}^{n} a_j x_j \right) = \sum_{j=1}^{n} a_j f(x_j)$$

for all $x_1, x_2, \cdots, x_n \in G$ and all rational numbers $a_1, a_2, \cdots, a_n$. The result follows from this equality.

The following theorem follows from Lemma 2.1, Lemma 2.2, Lemma 2.4, and Lemma 2.5.

Theorem 2.6 The functional equation $E_i f : G \rightarrow V$ is a quadratic-additive type functional equation, where $i = 1, 2, 3, \cdots, 18$.

3 Quadratic-additive mappings

Throughout this section, let $Y$ be a real normed space.

Theorem 3.1 Let $\varphi : G \setminus \{0\} \rightarrow [0, \infty)$ be a function satisfying one of the following conditions

1. \[ \lim_{n \to \infty} \frac{\varphi \left( 2^n x \right)}{2^n} = 0, \]
2. \[ \lim_{n \to \infty} 2^n \varphi \left( \frac{x}{2^n} \right) = 0 = \lim_{n \to \infty} \frac{\varphi \left( 2^n x \right)}{4^n}, \]
3. \[ \lim_{n \to \infty} 4^n \varphi \left( \frac{x}{2^n} \right) = 0 \]

for all $x \in X$. Let $f : G \rightarrow Y$ be a given mapping. If there exists a quadratic-additive mapping $F : G \rightarrow Y$ such that

$$\| f(x) - F(x) \| \leq \varphi(x)$$

for all $x \in G \setminus \{0\}$, then $F$ is a unique quadratic-additive mapping satisfying the inequality (4).

Proof. Assume that $F$ and $F'$ are two quadratic-additive mappings satisfying (4) for a given mapping $f : G \rightarrow Y$. Then there are additive mappings $A, A'$
and quadratic mappings $Q, Q'$ such that $F = Q + A$ and $F' = Q' + A'$.

For the case $\varphi : G \to [0, \infty]$ satisfies the condition (1) or (2), we have

$$\|Q(x) - Q'(x)\| = \frac{1}{4^n} \|Q(2^n x) - Q'(2^n x)\|$$

$$= \frac{1}{4^n} \|F(2^n x) - A(2^n x) - f(2^n x) + f(2^n x) + A'(2^n x) - F'(2^n x)\|$$

$$\leq \frac{1}{4^n} \|F(2^n x) - f(2^n x)\| + \frac{1}{4^n} \|A'(2^n x) - A(2^n x)\|$$

$$+ \frac{1}{4^n} \|f(2^n x) - F'(2^n x)\|$$

$$\leq \frac{2}{4^n} \varphi(2^n x) + \frac{1}{2^n} \|A'(x) - A(x)\|$$

for all $x \in G \setminus \{0\}$ and all $n \in N$. So we obtain the equality $Q(x) = Q'(x)$ for all $x \in G \setminus \{0\}$ by taking the limit of the above inequality as $n \to \infty$. If $\varphi : G \to [0, \infty]$ satisfies the condition (1), we get

$$\|A(x) - A'(x)\| = \frac{1}{2^n} \|A(2^n x) - A'(2^n x)\|$$

$$= \frac{1}{2^n} \|F(2^n x) - f(2^n x) + f(2^n x) - F'(2^n x)\|$$

$$\leq \frac{1}{2^n} \|F(2^n x) - f(2^n x)\| + \frac{1}{2^n} \|f(2^n x) - F'(2^n x)\|$$

$$\leq \frac{2}{2^n} \varphi(2^n x)$$

for all $n \in N$. So we obtain the equality $A(x) = A'(x)$ for all $x \in G \setminus \{0\}$ by taking the limit of the above inequality as $n \to \infty$. Therefore we get $F(x) = F'(x)$ for all $x \in G \setminus \{0\}$.

On the other hand if $\varphi : G \to [0, \infty]$ satisfies the condition (2), then we have

$$\|A(x) - A'(x)\| = 2^n \|A\left(\frac{x}{2^n}\right) - A'\left(\frac{x}{2^n}\right)\|$$

$$= 2^n \|F\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) + f\left(\frac{x}{2^n}\right) - F'\left(\frac{x}{2^n}\right)\|$$

$$\leq 2^n \|F\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\| + 2^n \|f\left(\frac{x}{2^n}\right) - F'\left(\frac{x}{2^n}\right)\|$$

$$\leq 2^{n+1} x \varphi\left(\frac{x}{2^n}\right)$$

for all $n \in N$. So we obtain the equality $A(x) = A'(x)$ for all $x \in G \setminus \{0\}$ by taking the limit of the above inequality as $n \to \infty$. Therefore we get $F(x) = F'(x)$ for all $x \in G \setminus \{0\}$.

For the case $\varphi : G \to [0, \infty]$ satisfies the condition (3), we have

$$\|A(x) - A'(x)\| = 2^n \|A\left(\frac{x}{2^n}\right) - A'\left(\frac{x}{2^n}\right)\|$$
Theorem 3.1 for the case (1), (2), or (3), respectively.

**Proof**

Inequality (5).

Since $F_n$ taking the limit of the above inequality as $n \to \infty$ for all $n \in \mathbb{N}$, we obtain the equality $(\ast)$ for any $n \in \mathbb{N}$, and for all $x \in G \backslash \{0\}$. Therefore we get $F(x) = F'(x)$ for all $x \in G \backslash \{0\}$. Since $F(0) = F'(0)$, we obtain the equality $F(x) = F'(x)$ for all $x \in G$ for any case.

**Corollary 3.2** Let $X$ be a normed space and let $p, k$ be real numbers with $p \neq 1, 2$ and $k \geq 0$. For a given mapping $f : X \to Y$, if there exists a quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq k\|x\|^p$$

for all $x \neq 0$, then $F$ is a unique quadratic-additive mapping satisfying the inequality (5).

**Proof.** Let $\varphi : X \to [0, \infty)$ be a mapping defined by $\varphi(x) = k\|x\|^p$ for all $x \in X \backslash \{0\}$. If $p < 1$, $1 < p < 2$, or $p > 2$, then this corollary follows from Theorem 3.1 for the case (1), (2), or (3), respectively.

**References**


Received: May 14, 2013