Some Properties of a Semi Dynamical System

Generated by von Forester-Losata Type

Partial Equations

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Abstract

In this paper we consider a semi dynamical system, \((T_t)_{t \geq 0}\) generated by different partial equations of Von Forester-Lasota a type. We investigate some of the properties of dynamical system in the space \(L^p\) with \(p < 1\).

Keywords: von Forester-Lasota equation, stability.

1-Introduction

First, we consider the partial differential equation

\[ \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \gamma u, \quad t \geq 0, \quad 0 \leq x \leq 1 \]  \hspace{1cm} (1)

with the initial condition

\[ u(0,x) = v(x), \quad 0 \leq x \leq 1 \]  \hspace{1cm} (2)
where \( v \) belong to some normed vector space \( V \) of functions defined on \([0,1]\).

The function \( \tilde{T}_t \) is given by the formula

\[
(\tilde{T}_t v)(x) = \tilde{u}(t, x) = e^{\gamma t} v(x e^{-t}), \quad x \in [0,1]
\]

(3)

where \( \tilde{u} \) is the unique solution of (1) and (2).

The second considered partial differential equation is

\[
\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \lambda(x)u , \quad t \geq 0 , \quad 0 \leq x \leq 1
\]

(4)

with the initial condition

\[
u(0,x) = v(x) , \quad 0 \leq x \leq 1
\]

(5)

where \( v \) belongs to some normed vector space \( V \) of functions defined on \([0,1]\) and \( \lambda: [0,1] \rightarrow \mathbb{R} \) is given continuous function. Let a semi dynamical system

\[
T_t: V \rightarrow V \quad \text{Be given by the formula} \quad (T_t v)(x) = u(t,x).
\]

It is clear that the unique of (4), (5) is given by the formula

\[
(T_t v)(x) = u(t,x) = e^{g(x)} e^{-g(x e^{-t})} v(x e^{-t}), \quad x \in [0,1]
\]

(6)

where

\[
g(x) = - \int_x^1 \frac{\lambda(s)}{s} ds \quad \text{with the condition}
\]

\[
\int_0^1 \frac{\lambda(s)}{s} ds = \infty.
\]

(7)

This can be found in Dawidowicz, Poskrobko (2006). There exists a connection between these two equations. It is easy to check that if \( u \) and \( \tilde{u} \) are the solutions of the equation (4) and (1), respectively, we have the equality

\[
\tilde{u}(t, x) = k(x) u(t, x),
\]

(8)

where

\[
k(x) = e^{\int_0^x \frac{\lambda(s)-\gamma}{s} ds} \quad \text{and} \quad \gamma = \lambda(0).
\]

All properties of the system \( (\tilde{T}_t)_{t \geq 0} \) and \( (T_t)_{t \geq 0} \) depend on the value of the constant \( \gamma = \lambda(0) \).
Definition 1.1 A function \( v_0 \in V \) is a periodic point of the semi group \((T_t)_{t \geq 0}\) with a period \( t_0 \geq 0 \) if and only if \( T_{t_0} v_0 = v_0 \). A number \( t_0 > 0 \) is called a principal period of a periodic point \( v_0 \) if and only if the set of all periods of \( v_0 \) is equal \( N t_0 \).

Definition 1.2 The semi group \((T_t)_{t \geq 0}\) is strongly stable in \( V \) iff for every \( v \in V \), \( \lim_{t \to \infty} T_t v = 0 \) in \( V \).

Definition 1.3 The semi group \((T_t)_{t \geq 0}\) is exponentially stable on \( V \) iff there exists \( D < \infty \) and \( \omega > 0 \) such that
\[
\|T_t\| \leq D e^{-\omega t}, \text{ for } t \geq 0
\]
where \( \|\cdot\| \) is the norm of \( V \).

2. Properties of dynamical system \((T_t)_{t \geq 0}\)

Theorem 2.1 Assume that

\[
\exists C, q > 0 \quad \forall x \in [0, 1] \quad |\lambda(0) - \lambda(x)| \leq C x^q
\] (9)

Holds, then we have the equivalence: the function \( u \) belongs to the space \( \hat{L}_p \) if and only if \( u \in \hat{L}_p, p < 1 \).

Proof: by (9), \( u \in L_p \) iff \( \hat{u} \in L_p \). This can be found in Dawidowicz, Poskrobko (2006). Assume that \( u \in \hat{L}_p \) we have

\[
S_{(0,\alpha)}(\hat{u}) = \sup_{x \in (0,\alpha)} \left( \frac{1}{x} \int_0^x |\hat{u}(t, s)|^p \, ds \right)^{\frac{1}{p}} = \sup_{x \in (0,\alpha)} \left( \frac{1}{x} \int_0^x |k(s)u(t, s)|^p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \sup_{x \in (0,\alpha)} \left( \frac{1}{x} \int_0^x e^{p f_0 \int_0^s \frac{1}{\sigma} |\lambda(0) - \lambda(\sigma)| \, d\sigma} |u(t, s)|^p \, ds \right)^{\frac{1}{p}}
\]

\[
|\lambda(0) - \lambda(\sigma)| \leq C x^q , \quad e^{pc f_0 \int_0^s \frac{1}{\sigma^q} \, d\sigma} = e^{pcq}\frac{1}{q}
\]

\[
\leq \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x e^{pcq}\frac{1}{q} |u(t, s)|^p \, ds \right)^{\frac{1}{p}} \leq e^{pcq} \sup_{x \in (0,\alpha)} \left( \frac{1}{x} \int_0^x |u(t, s)|^p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq e^{pcq} S_{(0,\alpha)}(u), \text{ So } \lim_{\alpha \to 0} S_{(0,\alpha)}(\hat{u}) = 0
\]

In this same manner we can establish the inverse implication.
\[
S_{(0,a)}(u) = \sup_{x \in (0,a)} \left( \frac{1}{x} \int_0^x |u(t,s)|^p \, ds \right)^{\frac{1}{p}} = \sup_{x \in (0,a)} \left( \frac{1}{x} \int_0^x \left| \frac{\tilde{u}(t,s)}{k(s)} \right|^p \, ds \right)^{\frac{1}{p}} \\
\leq \sup_{x \in (0,a)} \left( \frac{1}{x} \int_0^x \left| \frac{\tilde{u}(t,s)}{C_{\alpha}^q} \right|^p \, ds \right)^{\frac{1}{p}} \\
\leq \sup_{x \in (0,a)} \left( \frac{1}{x} \int_0^x e^{-\frac{pC_{\alpha}^q}{q} \left| \tilde{u}(t,s) \right|} \, ds \right)^{\frac{1}{p}} \\
\leq e^{-\frac{pC_{\alpha}^q}{q} \tilde{S}_{(0,a)}(\tilde{u})}, \quad \lim_{a \to 0} S_{(0,a)}(u) = 0 .
\]

**Theorem 2.2** If $\lambda(0) > 0$, then for any $t$, there exists such $v_0 \in \tilde{L}_p$, $p < 1$ that

\[ T_{t,v_0} = v_0 \quad (10) \]

Moreover,

\[ T_{t,v_0} = v_0 \text{ if and only if } t = nt_0 \text{ for some positive integer } (11) \]

**Proof:** Let $\omega$ be an arbitrary function belong to $L_p$, defined on the interval $[e^{-t_0}, 1]$ and satisfying the following conditions:

\[ e^{-g(e^{-t_0})} \omega(e^{-t_0}) = \omega(1), \quad (12) \]

\[ e^{-g(e^{-t})} \omega(e^{-t}) \neq \omega(1) \quad \forall t \in (0, t_0). \quad (13) \]

Consider the function $v$ on the interval $(0, 1)$

\[ v(x) = e^{g(x)} e^{-g(x e^{nt_0})} \omega(x e^{nt_0}) \quad \text{for } x \in [e^{-(n+1)t_0}, e^{-nt_0}]. \]

The function $v$ is defining on the whole interval $(0, 1) = \bigcup_{n=0}^{\infty} (e^{-(n+1)t_0}, e^{-nt_0})$ and comes into being squeezing the graph of the function $\omega$ into each of the intervals

\[ (e^{-(n+1)t_0}, e^{-nt_0}). \]

By assumption of the continuity of $\omega$ on $[e^{-t_0}, 1]$ follows its boundedness, i.e.

\[ \exists M > 0 \text{ such that } |\omega(x)| \leq M \text{ for each } x \in [e^{-t_0}, 1]. \]

By the above for $x \in [e^{-(n+1)t_0}, e^{-nt_0}]$, we have the estimation

\[ |v(x)| = e^{g(x)} e^{-g(x e^{nt_0})} |\omega(x e^{nt_0})| \leq Me^{g(x)} \sup_{x \in [e^{-g(x)_1}, e^{-g(x)}]} |e^{-g(x)}| \leq M_1 e^{g(x)} \]

where $M_1 = M \sup_{x \in [e^{-t_0}, 1]} e^{-g(x)}$, from the assumption (7) $\lim_{x \to 0} e^{g(x)} = 0$ so we deduce that $v(0) = 0$. we obtain the continuous function $v$ defined on the whole interval $[0,1]$. The property (10) follows from (12), while the property (11) from (13). Our next goal is to show that $v \in \tilde{L}_p$. Under theorem (4.1) [6] we know that $\tilde{v} \in \tilde{L}_p$ for
\( \gamma > 0 \), where \( \bar{v} \) is the solution of the equation (1). It clearly the same conclusion for the function \( v \) by theorem (2.1)

**Theorem 2.3** if \( \lambda(0) > 0 \) then the set of periodic points of (4) is dense in \( \hat{L}_p, p < 1 \).

**Proof:** let \( \varepsilon > 0 \) and let \( v \in \hat{L}_p \). let \( v \) be a periodic solution of (4) Defined by the formula (6)

Since \( v \) and \( \omega \) belong to \( \hat{L}_p \) there exist to such that \( S_{(0, e^{-t_1})}(v) < \frac{\varepsilon}{4} \) and \( S_{(0, e^{-t_1})}(\omega) < \frac{\varepsilon}{4} \).

We know that \( v(x) = \frac{\bar{v}(x)}{k(x)} \), where \( \bar{v} \) is the periodic solution of (1)

The assumption \( \lambda(0) > 0 \), guarantees the density of the set of periodic points of (1), so \( S_{(0,1)}(v - \bar{v}) < \frac{\varepsilon}{4} \) and \( S_{(0,1)}(\omega - \bar{v}) < \frac{\varepsilon}{4} \)

Thus

\[
S_{(0,1)}(v - \omega) \leq S_{(0, e^{-t_1})}(v - \omega) + S_{(e^{-t_1})}(v) \leq S_{(0, e^{-t_1})}(v) + S_{(0, e^{-t_1})}(v + \bar{v} - \omega) \leq S_{(0, e^{-t_1})}(v) + S_{(0, e^{-t_1})}(\omega) + S_{(0,1)}(v - \bar{v}) + S_{(0,1)}(\bar{v} - \omega) < \varepsilon.
\]

**Theorem 2.4** if \( \lambda(0) \leq 0 \) then for every \( v \in \hat{L}_p \), \( \lim_{t \to \infty} S_{(0,1)}(T_t v) = 0 \).

Moreover, if \( \lambda(0) < 0 \), the semigroup \( (T_t)_{t \geq 0} \) is exponentially stable.

**Proof:** Take any \( v \in \hat{L}_p \), then we have

\[
S_{(0,1)}(T_t v) = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x |u(t, s)|^p ds \right)^{\frac{1}{p}} = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x \left| \frac{1}{k(s)} (T_t \bar{v})(s) \right|^p ds \right)^{\frac{1}{p}}
\]

And, \( (s) = e^{\int_0^s \sigma \lambda(0) - \lambda(\sigma) d\sigma} \),

so \( k(s) = e^{c \frac{s^q}{q}} \)

Then

\[
S_{(0,1)}(T_t v) = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x \left| e^{c \frac{s^q}{q}} (T_t \bar{v})(s) \right|^p ds \right)^{\frac{1}{p}}
\]

Since \( \sup_{x \in (0,1)} e^{c \frac{s^q}{q}} \leq e^c \)

then

\[
S_{(0,1)}(T_t v) \leq e^c S_{(0,1)}(T_t \bar{v})
\]
Applying theorem (4.3)[1] we can assert that $S_{(0,1)}(T_t \nu) \to 0$, as $t \to \infty$. This proves the first part of the theorem. The second one follows immediately from the same above inequality and theorem (4.3)[6] with $D = e^{\xi t}$ and $\omega = -\lambda(0)$

$$S_{(0,1)}(T_t \nu) \leq e^{\xi t}S_{(0,1)}(T_t \vartheta) \leq e^{\xi t}e^{\lambda(0)t} = e^{\xi t+\lambda(0)t}.$$

References


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