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On the Distribution of Zeros of Polynomials and Analytic Functions

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Abstract

This paper presents several results on the distribution of zeros of polynomials and analytic functions. The results of this paper include some existing ones as special cases.

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1 Introduction

The Eneström-Kakeya theorem [7] given below is classical and well known in the theory of zero distribution of polynomials.

Theorem A. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be an nth-order polynomial such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then all the zeros of $P(z)$ lie in the disk $|z| \leq 1$.

Many attempts have been made in the literature to generalize the Eneström-Kakeya theorem (see [1], [2], [3] and references cited therein). Among them, Aziz and Mohammad [1] extended the Eneström-Kakeya theorem in the following way.

Theorem B. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be an nth-order polynomial with positive real coefficients such that for some $t_1 > t_2 \geq 0$*

$$t_1 t_2 a_i + (t_1 - t_2) a_{i-1} - a_{i-2} \geq 0, \quad i = 1, 2, \cdots, n+1,$$

where $a_i = 0$ if $i < 0$ or $i > n$. Then all the zeros of $P(z)$ lie in the disk $|z| \leq t_1$.

Theorem B reduces to the Eneström-Kakeya theorem if $t_1 = 1$ and $t_2 = 0$. Some extensions of Theorem B were made in [4], [5], [6]. Aziz and Zagar [2] relaxed the coefficient conditions in the Eneström-Kakeya theorem and obtained the following result.

Theorem C. *If $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order polynomial with positive real coefficients such that*

$$t^2 a_i \geq a_{i-2}, \quad i = 2, 3, \cdots, n,$$

then $P(z)$ has all its zeros in the disk

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq t + \frac{a_{n-1}}{a_n}.$$

Aziz and Mohammad [1] also studied the zeros of analytic functions and proved the following theorem.

Theorem D. *Let $P(z) = \sum_{i=0}^{\infty} a_i z^i$ be analytic in $|z| < t$. If*

$$a_i > 0, \quad i = 0, 1, 2, \cdots, \quad \text{and} \quad a_i - t a_{i+1} \geq 0, \quad i = 0, 1, 2, \cdots,$$

then $P(z)$ does not vanish in the disk $|z| < t$.

The work of this paper was inspired by Theorems B and C. For example it is natural to ask what happens if the coefficient conditions of Theorems B are imposed respectively on the coefficients of even powers and odd powers of the variable as in Theorem C. Another interesting question is what happens if the coefficient conditions of Theorem B are satisfied for analytic functions. Some answers to such questions are given in the next section.

2 Theorems and proofs

Theorem 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be an nth-order polynomial with positive real coefficients such that for some* $t_1 > t_2 \geq 0$

$$t_1 t_2 a_i + (t_1 - t_2) a_{i-2} - a_{i-4}, \quad i = 2, 3, \cdots, n+2,$$

where $a_i = 0$ *if* $i < 0$ *or* $i > n$. *Then all the zeros of* $P(z)$ *lie in the disk*

$$\Big| z + \frac{a_{n-1}}{a_n} \Big| \leq \sqrt{t_1} + \frac{a_{n-1}}{a_n}.$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (t_2 + z^2)(t_1 - z^2) P(z) \\ &= \{t_1 t_2 + (t_1 - t_2) z^2 - z^4\} P(z) \\ &= -a_n z^{n+4} - a_{n-1} z^{n+3} + \phi(z), \end{aligned}$$

where

$$\begin{aligned} \phi(z) &= \{(t_1 - t_2) a_n - a_{n-2}\} z^{n+2} + \{(t_1 - t_2) a_{n-1} - a_{n-3}\} z^{n+1} \\ &\quad + \sum_{i=4}^{n} \{t_1 t_2 a_i + (t_1 - t_2) a_{i-2} - a_{i-4}\} z^i \\ &\quad + \{t_1 t_2 a_3 + (t_1 - t_2) a_1\} z^3 + \{t_1 t_2 a_2 + (t_1 - t_2) a_0\} z^2 \\ &\quad + t_1 t_2 a_1 z + t_1 t_2 a_0. \end{aligned}$$

If $|z| > \sqrt{t_1}$, then

$$\begin{aligned} |\phi(z)| &\leq |z|^{n+3} \bigg\{ \frac{(t_1 - t_2) a_n - a_{n-2}}{|z|} + \frac{(t_1 - t_2) a_{n-1} - a_{n-3}}{|z|^2} \\ &\quad + \sum_{i=4}^{n} \frac{t_1 t_2 a_i + (t_1 - t_2) a_{i-2} - a_{i-4}}{z^{n+3-i}} \\ &\quad + \frac{t_1 t_2 a_3 + (t_1 - t_2) a_1}{|z|^n} + \frac{t_1 t_2 a_2 + (t_1 - t_2) a_0}{|z|^{n+1}} \\ &\quad + \frac{t_1 t_2 a_1}{|z|^{n+2}} + \frac{t_1 t_2 a_0}{|z|^{n+3}} \bigg\} \\ &\leq |z|^{n+3} \bigg\{ \frac{(t_1 - t_2) a_n - a_{n-2}}{\sqrt{t_1}} + \frac{(t_1 - t_2) a_{n-1} - a_{n-3}}{\sqrt{t_1}^2} \\ &\quad + \sum_{i=4}^{n} \frac{t_1 t_2 a_i + (t_1 - t_2) a_{i-2} - a_{i-4}}{\sqrt{t_1}^{n+3-i}} \\ &\quad + \frac{t_1 t_2 a_3 + (t_1 - t_2) a_1}{\sqrt{t_1}^n} + \frac{t_1 t_2 a_2 + (t_1 - t_2) a_0}{\sqrt{t_1}^{n+1}} \\ &\quad + \frac{t_1 t_2 a_1}{\sqrt{t_1}^{n+2}} + \frac{t_1 t_2 a_0}{\sqrt{t_1}^{n+3}} \bigg\} \\ &= |z|^{n+3} (\sqrt{t_1} a_n + a_{n-1}). \end{aligned}$$

Then, for $|z| > \sqrt{t_1}$, we have

$$\begin{aligned} |\Phi(z)| &\geq |z|^{n+3}|a_n z + a_{n-1}| - |\phi(z)| \\ &\geq |z|^{n+3}\{|a_n z + a_{n-1}| - (\sqrt{t_1}a_n + a_{n-1})\} \\ &> 0, \end{aligned}$$

if

$$|a_n z + a_{n-1}| > \sqrt{t_1}a_n + a_{n-1}.$$

Hence all the zeros of $\Phi(z)$ with modulus greater than $\sqrt{t_1}$ lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n}\right| \leq \sqrt{t_1} + \frac{a_{n-1}}{a_n}.$$

Since the zeros of $\Phi(z)$ with modulus less than or equal to $\sqrt{t_1}$ are already contained in the above disk, the proof is completed.

If $t_2 = 0$, Theorem 1 reduces to Theorem C.

Theorem 2. *Let $P(z) = \sum_{i=0}^{\infty} a_i z^i$, where $a_i > 0$, $i = 0, 1, 2, \cdots$. Suppose that for some $t_1 > t_2 \geq 0$*

$$t_1 t_2 a_{i-2} + (t_1 - t_2)a_{i-1} - a_i \geq 0, \quad i = 1, 2, 3, \cdots,$$

with $a_{-1} = 0$, and that $P(z)$ is analytic in $|z| < 1/t_1$. Then $P(z)$ does not vanish in the disk $|z| < 1/t_1$.

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (1 + t_2 z)(1 - t_1 z)P(z) \\ &= \{1 - (t_1 - t_2)z - t_1 t_2 z^2\}P(z) \\ &= a_0 + \phi(z), \end{aligned}$$

where

$$\phi(z) = \{a_1 - (t_1 - t_{2)}a_0\}z + \sum_{i=2}^{\infty}\{a_i - (t_1 - t_2)a_{i-1} - t_1 t_2 a_{i-2}\}z^i.$$

If $|z| < 1/t_1$, then

$$\begin{aligned} |\phi(z)| &\leq |\{a_1 - (t_1 - t_2)a_0\}||z| + \sum_{i=2}^{\infty}|\{a_i - (t_1 - t_2)a_{i-1} - t_1 t_2 a_{i-2}\}||z|^i \\ &\leq \frac{(t_1 - t_2)a_0 - a_1}{t_1} + \sum_{i=2}^{\infty}\frac{t_1 t_2 a_{i-2} + (t_1 - t_2)a_{i-1} - a_i}{t_1^i} \\ &= a_0. \end{aligned}$$

Since $\phi(0) = 0$, it follows by Schwarz lemma that

$$|\phi(z)| \leq t_1 a_0 |z| \quad \text{for} \quad |z| < 1/t_1.$$

Then, for $|z| < 1/t_1$

$$\begin{aligned} |\Phi(z)| &\geq a_0 - |\phi(z)| \\ &\geq a_0 - t_1 a_0 |z| \\ &> 0, \end{aligned}$$

if $|z| < 1/t_1$. Hence $\Phi(z)$ ($P(z)$ also) does not vanish in the disk $|z| < 1/t_1$, and the proof is completed.

If $t_2 = 0$, Theorem 2 reduces to Theorem D.

Theorem 3. *Let $P(z) = \sum_{i=0}^{\infty} a_i z^i$ be analytic in $|z| < t_1$. If $a_i > 0$, $i = 0, 1, 2, \cdots$, and if for some $t_1 \geq t_2 > 0$*

$$a_i - (t_1 - t_2)a_{i+1} - t_1 t_2 a_{i+2} \geq 0, \quad i = 0, 1, 2, \cdots,$$

then $P(z)$ does not vanish in the disk

$$\left| z - \frac{M^2 \alpha_0 \alpha_1}{1 - M^2 \alpha_1^2} \right| < \frac{M \alpha_0}{1 - M^2 \alpha_1^2},$$

where

$$\begin{aligned} \alpha_0 &= t_1 t_2 a_0, \\ \alpha_1 &= t_1 t_2 a_1 + (t_1 - t_2) a_0, \\ M &= \frac{1}{t_1(a_0 + t_2 a_1)}. \end{aligned}$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (t_2 + z)(t_1 - z)P(z) \\ &= \{t_1 t_2 + (t_1 - t_2)z - z^2\} P(z) \\ &= t_1 t_2 a_0 + \{t_1 t_2 a_1 + (t_1 - t_2) a_0\} z + \phi(z), \end{aligned}$$

where

$$\phi(z) = \sum_{i=2}^{\infty} \{t_1 t_2 a_i + (t_1 - t_2) a_{i-1} - a_i\} z^i.$$

If $|z| < t_1$, then

$$\begin{aligned} |\phi(z)| &\leq \sum_{i=2}^{\infty} |\{t_1 t_2 a_i + (t_1 - t_2) a_{i-1} - a_{i-2}\} z^i| \\ &\leq \sum_{i=2}^{\infty} \{a_{i-2} - (t_1 - t_2) a_{i-1} - t_1 t_2 a_i\} t_1^i \\ &= t_1^2 (a_0 + t_2 a_1). \end{aligned}$$

Since $\phi(0) = 0$, it follows by Schwarz lemma that

$$|\phi(z)| \le t_1(a_0 + t_2 a_1)|z| \quad \text{for} \quad |z| < t_1.$$

Then, for $|z| < t_1$

$$\begin{aligned} |\Phi(z)| &\ge |t_1 t_2 a_0 + \{t_1 t_2 a_1 + (t_1 - t_2)a_0\}z| - |\phi(z)| \\ &\ge |t_1 t_2 a_0 + \{t_1 t_2 a_1 + (t_1 - t_2)a_0\}z| - t_1(a_0 + t_2 a_1)|z| \\ &> 0, \end{aligned}$$

if

$$|z| < M|\alpha_0 + \alpha_1 z|,$$

where

$$\begin{aligned} \alpha_0 &= t_1 t_2 a_0, \\ \alpha_1 &= t_1 t_2 a_1 + (t_1 - t_2)a_0, \\ M &= \frac{1}{t_1(a_0 + t_2 a_1)}. \end{aligned}$$

It is easy to show that the region defined by

$$|z| < M|\alpha_0 + \alpha_1 z|,$$

is the disk

$$\left| z - \frac{M^2 \alpha_0 \alpha_1}{1 - M^2 \alpha_1^2} \right| < \frac{M \alpha_0}{1 - M^2 \alpha_1^2}.$$

Since t_1 and $-t_2$ are not contained in the above disk, the proof is completed.

Setting $t_1 = t_2$ in Theorem 3, we obtain the following corollary.

Corollary 4. *Let $P(z) = \sum_{i=0}^{\infty} a_i z^i$ be analytic in $|z| < t$. If*

$$a_i > 0, \ \ i = 0, 1, 2, \cdots, \quad \text{and} \quad a_i - t^2 a_{i+2} \ge 0, \quad i = 0, 1, 2, \cdots,$$

then $P(z)$ does not vanish in the disk

$$\left| z - \frac{t^2 a_1}{a_0 + 2t a_1} \right| < \frac{t(a_0 + t a_1)}{a_0 + 2t a_1}.$$

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