Faintly \((m, \mu)\)-Continuous Functions

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Abstract

A new class of functions, called faintly \((m, \mu)\)-continuous functions, has been defined and studied. Some characterizations and several properties concerning faintly \((m, \mu)\)-continuous functions are obtained. The relationships between faintly \((m, \mu)\)-continuous functions and other related generalized forms of \((m, \mu)\)-continuity are also discussed.

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1 Introduction

In 1982, P. E. Long and L. L. Herrington [5] introduced the notion of faintly continuous functions. Some properties of faintly continuous functions are studied in [8, 11]. In 1990, the present authors [10] introduced and investigated three weaker forms of faint continuity, that is, faint semi-continuity, faint pre-continuity and faint \(\beta\)-continuity. Recently, S. Jafari and T. Noiri [4] have introduced and investigated the notion of faintly \(\alpha\)-continuous functions. A. A. Nasef [7] has also introduced weaker forms of faint continuity, that is, faint \(\alpha\)-continuity and faint \(\gamma\)-continuity (or faint \(b\)-continuity). In 2004, T. Noiri and V. Popa [9] introduced the notion of faintly \(m\)-continuous functions and investigated their properties and the relationships between faint \(m\)-continuity
and other related generalized forms of continuity. On the other hand, the present authors introduced the notions of \((m, \mu)\)-continuous functions, almost \((m, \mu)\)-continuous functions and weakly \((m, \mu)\)-continuous functions [12].

In this paper, we introduce the notion of faintly \((m, \mu)\)-continuous functions as functions from a set \(X\) satisfying some minimal conditions into a generalized topological spaces and investigate their properties and the relationships between faint \((m, \mu)\)-continuity and other related generalized forms of \((m, \mu)\)-continuity.

2 Preliminaries

We recall some notions defined in [2]. Let \(X\) be a nonempty set and \(\text{exp} \ X\) the power set of \(X\). We call a class \(\mu \subseteq \text{exp} \ X\) a generalized topology (briefly, GT) if \(\emptyset \in \mu\) and the arbitrary union of elements of \(\mu\) belongs to \(\mu\). A set \(X\) with a GT \(\mu\) on it is said to be a generalized topological space (briefly, GTS) and is denoted by \((X, \mu)\). For a GTS \((X, \mu)\), the elements of \(\mu\) are called \(\mu\)-open sets and the complements of \(\mu\)-open sets are called \(\mu\)-closed sets. For \(A \subseteq X\), we denote by \(c_\mu(A)\) the intersection of all \(\mu\)-closed sets containing \(A\) and by \(i_\mu(A)\) the union of all \(\mu\)-open sets contained in \(A\). Then, we have \(i_\mu(c_\mu(A)) = c_\mu(A)\) and \(X - i_\mu(A) = c_\mu(X - A)\). According to [3], for \(A \subseteq X\) and \(x \in X\), we have \(x \in c_\mu(A)\) if \(x \in M \in \mu\) implies \(M \cap A \neq \emptyset\). A subset \(R\) of a generalized topological space \((X, \mu)\) is said to be \(\mu_r\)-open [3] (resp. \(\mu_r\)-closed) if \(R = i_\mu(c_\mu(R))\) (resp. \(R = c_\mu(i_\mu(R))\)).

**Definition 2.1.** [14] A subfamily \(m_X\) of the power set \(\text{P}(X)\) of a nonempty set \(X\) is called a minimal structure (briefly, \(m\)-structure) on \(X\) if \(\emptyset \in m_X\) and \(X \in m_X\).

By \((X, m_X)\) (briefly, \((X, m)\)), we denote a nonempty set of \(X\) with a minimal structure \(m_X\) on \(X\) and call it an \(m\)-space. Each member of \(m_X\) is said to be \(m_X\)-open (briefly, \(m\)-open) and the complement of an \(m_X\)-open set is said to be \(m_X\)-closed (briefly, \(m\)-closed).

**Definition 2.2.** [6] Let \(X\) be a nonempty set and \(m_X\) a minimal structure on \(X\). For a subset \(A\) of \(X\), the \(m_X\)-closure of \(A\) and the \(m_X\)-interior of \(A\) are defined as follows:

1. \(m_X Cl(A) = \cap \{F : A \subseteq F, X - F \in m_X\}\);
2. \(m_X Int(A) = \cup \{U : U \subseteq A, U \in m_X\}\).

**Lemma 2.3.** [6] Let \(X\) be a nonempty set and \(m_X\) be an \(m\)-structure on \(X\). For subset \(A\) and \(B\) of \(X\), the following properties hold:

1. \(m_X Cl(X - A) = X - m_X Int(A)\) and \(m_X Int(X - A) = X - m_X Cl(A)\);
Let $(X, \mu)$ be a generalized topological space and $A$ be a subset of $X$. Then a point $x \in X$ is called a $\zeta$-cluster point of $A$ if $c_{\mu}(V) \cap A \neq \emptyset$ for every $\mu$-open set $V$ containing $x$. The set of all $\zeta$-cluster points of $A$ is called the $\zeta$-closure of $A$ and is denoted by $c_\zeta(A)$. A set $A$ is said to be $\zeta$-closed if $A = c_\zeta(A)$. The complement of a $\zeta$-closed set is said to be $\zeta$-open.

The union of all $\zeta$-open sets contained in a subset $A$ is called the $\zeta$-interior of $A$ and denoted by $i_\zeta(A)$. The set of all $\zeta$-open sets in $(X, \mu)$ is denoted by $\mu_\zeta$.

**Lemma 3.2.** Let $(X, \mu)$ be a generalized topological space. A subset $A$ of $X$ is $\zeta$-open if and only if for every $x \in A$, there exists a $\mu$-open set $U$ such that $x \in U \subseteq c_{\mu}(U) \subseteq A$.

**Proof.** Let $A$ be $\zeta$-open and $x \in A$. Then $X - A$ is $\zeta$-closed and $c_\zeta(X - A) = X - A$. There exists a $\mu$-open set $U$ containing $x$ such that $c_{\mu}(U) \cap (X - A) = \emptyset$. Hence, $c_{\mu}(U) \subseteq A$.

Conversely, suppose that $x \notin X - A$. Then $x \in A$ and by the hypothesis there exists a $\mu$-open set $U$ such that $x \in U \subseteq c_{\mu}(U) \subseteq A$. Then $c_{\mu}(U) \cap (X - A) = \emptyset$ and hence $x \notin c_\zeta(X - A)$. Therefore, $X - A$ is $\zeta$-closed. Hence, $A$ is $\zeta$-open. \qed
**Definition 3.3.** Let \((X, m_X)\) be an \(m\)-space and \((Y, \mu)\) be a generalized topological space. A function \(f : (X, m_X) \rightarrow (Y, \mu)\) is said to be faintly \((m, \mu)\)-continuous at \(x \in X\) if for each \(\zeta\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists \(U \in m_X\) containing \(x\) such that \(f(U) \subseteq V\). The function \(f\) is said to be faintly \((m, \mu)\)-continuous if it is faintly \((m, \mu)\)-continuous at each point \(x \in X\).

**Theorem 3.4.** A function \(f : (X, m_X) \rightarrow (Y, \mu)\) is faintly \((m, \mu)\)-continuous at \(x \in X\) if and only if for each \(\zeta\)-open set \(V\) of \(Y\) containing \(f(x)\), \(x \in m_X \text{Int}(f^{-1}(V))\).

**Proof.** Let \(f\) be faintly \((m, \mu)\)-continuous at \(x \in X\) and \(V\) be a \(\zeta\)-open set of \(Y\) containing \(f(x)\). Then, there exists \(U \in m_X\) containing \(x\) such that \(f(U) \subseteq V\). Then we have \(x \in U \subseteq f^{-1}(V)\) and hence \(x \in m_X \text{Int}(f^{-1}(V))\).

Conversely, let \(V\) be a \(\zeta\)-open set containing \(f(x)\). Then by the hypothesis we have \(x \in m_X \text{Int}(f^{-1}(V))\). There exists \(U \in m_X\) containing \(x\) such that \(x \in U \subseteq f^{-1}(V)\); hence \(f(U) \subseteq V\). This shows that \(f\) is faintly \((m, \mu)\)-continuous at \(x \in X\).

**Theorem 3.5.** For a function \(f : (X, m_X) \rightarrow (Y, \mu)\), the following properties are equivalent:

1. \(f\) is faintly \((m, \mu)\)-continuous;
2. \(f^{-1}(V) = m_X \text{Int}(f^{-1}(V))\) for every \(\zeta\)-open set \(V\) of \(Y\);
3. \(f^{-1}(F) = m_X \text{Cl}(f^{-1}(F))\) for every \(\zeta\)-closed set \(F\) of \(Y\).

**Proof.** (1)\(\Rightarrow\)(2) Let \(V\) be a \(\zeta\)-open set of \(Y\) and \(x \in f^{-1}(V)\). Then \(f(x) \in V\), there exists \(U \in m_X\) containing \(x\) such that \(f(U) \subseteq V\). It follows that \(x \in U \subseteq f^{-1}(V)\). Hence, \(x \in m_X \text{Int}(f^{-1}(V))\). Therefore, \(f^{-1}(V) \subseteq m_X \text{Int}(f^{-1}(V))\) and hence \(f^{-1}(V) = m_X \text{Int}(f^{-1}(V))\).

(2)\(\Rightarrow\)(3) Let \(F\) be a \(\zeta\)-closed set of \(Y\). Then by (2), we have \(X - f^{-1}(F) = f^{-1}(Y - F) = m_X \text{Int}(f^{-1}(Y - F)) = X - m_X \text{Cl}(f^{-1}(F))\). Hence, \(f^{-1}(F) = m_X \text{Cl}(f^{-1}(F))\).

(3)\(\Rightarrow\)(1) Let \(V\) be a \(\zeta\)-open set of \(Y\) containing \(f(x)\). By (3), we have \(X - f^{-1}(V) = f^{-1}(Y - V) = m_X \text{Cl}(f^{-1}(Y - V)) = X - m_X \text{Int}(f^{-1}(V))\). Therefore, \(f^{-1}(V) = m_X \text{Int}(f^{-1}(V))\). Since \(x \in f^{-1}(V) = m_X \text{Int}(f^{-1}(V))\), there exists \(U \in m_X\) containing \(x\) such that \(x \in U \subseteq f^{-1}(V)\). Hence, \(f(U) \subseteq V\). Therefore, \(f\) is faintly \((m, \mu)\)-continuous.

**Definition 3.6.** [12] Let \((X, m_X)\) be an \(m\)-space and \((Y, \mu)\) be a generalized topological space. A function \(f : (X, m_X) \rightarrow (Y, \mu)\) is said to be \((m, \mu)\)-continuous at \(x \in X\) if for each \(\mu\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists \(U \in m_X\) containing \(x\) such that \(f(U) \subseteq V\). The function \(f\) is said to be \((m, \mu)\)-continuous if it is \((m, \mu)\)-continuous at each point \(x \in X\).
Lemma 3.7. Let \((X, \mu)\) be a generalized topological space. Then \(\mu_\zeta\) is a generalized topology on \(X\).

Proof. \(\emptyset \in \zeta_\mu\) is evident. Let \(G_k\) be \(\zeta\)-open for all \(k \in K\). Let \(x \in \bigcup_{k \in K} G_k\). Then \(x \in G_{k_0}\) for some \(k_0 \in K\). There exists \(U \in \mu\) such that \(x \in U \subseteq c_\mu(U) \subseteq G_{k_0} \subseteq \bigcup_{k \in K} G_k\). Hence, \(\bigcup_{k \in K} G_k \in \zeta_\mu\). \(\square\)

Theorem 3.8. For a function \(f : (X, m_X) \to (Y, \mu)\), the following properties are equivalent:

1. \(f\) is faintly \((m, \mu)\)-continuous;
2. \(f : (X, m_X) \to (Y, \mu_\zeta)\) is \((m, \mu)\)-continuous;
3. \(f^{-1}(V) = m_X \text{Int}(f^{-1}(V))\) for every \(\zeta\)-open set \(V\) of \(Y\);
4. \(f^{-1}(F) = m_X \text{Cl}(f^{-1}(F))\) for every \(\zeta\)-closed set \(F\) of \(Y\).

Proof. The proof follows from Definition 3.3 and 3.6 and Theorem 3.5. \(\square\)

Corollary 3.9. Let \(m_X\) have property B. For a function \(f : (X, m_X) \to (Y, \mu)\), the following properties are equivalent:

1. \(f\) is faintly \((m, \mu)\)-continuous;
2. \(f : (X, m_X) \to (Y, \mu_\zeta)\) is \((m, \mu)\)-continuous;
3. \(f^{-1}(V)\) is \(m_X\)-open in \(X\) for every \(\zeta\)-open set \(V\) of \(Y\);
4. \(f^{-1}(F)\) is \(m_X\)-closed in \(X\) for every \(\zeta\)-closed set \(F\) of \(Y\).

Proof. The proof follows from Theorem 3.8 and Lemma 2.5. \(\square\)

Theorem 3.10. Let \(m_X\) have property B. For a function \(f : (X, m_X) \to (Y, \mu)\), the following statements are equivalent:

1. \(f\) is faintly \((m, \mu)\)-continuous;
2. \(f^{-1}(V)\) is \(m_X\)-open in \(X\) for every \(\zeta\)-open set \(V\) of \(Y\);
3. \(f^{-1}(F)\) is \(m_X\)-closed in \(X\) for every \(\zeta\)-closed set \(F\) of \(Y\);
4. \(f : (X, m_X) \to (Y, \mu_\zeta)\) is \((m, \mu)\)-continuous;
5. \(m_X \text{Cl}(f^{-1}(A)) \subseteq f^{-1}(c_\zeta(A))\) for every subset \(A\) of \(Y\);
6. \(f^{-1}(i_\zeta(B)) \subseteq m_X \text{Int}(f^{-1}(B))\) for every subset \(B\) of \(Y\).
Proof. (1)⇒(2) Let \( V \) be a \( \zeta \)-open set of \( Y \) and \( x \in f^{-1}(V) \). Since \( f(x) \in V \) and \( f \) is faintly \((m,\mu)\)-continuous, there exists \( U \in m_X \) such that \( f(U) \subseteq V \). Thus \( x \in U \subseteq f^{-1}(V) \), then \( x \in m_X \text{Int}(f^{-1}(V)) \). Therefore, \( f^{-1}(V) \subseteq m_X \text{Int}(f^{-1}(V)) \). Consequently, \( f^{-1}(V) = m_X \text{Int}(f^{-1}(V)) \) and hence \( f^{-1}(V) \) is \( m_X \)-open in \( X \).

(2)⇒(1) Let \( x \in X \) and \( V \) be a \( \zeta \)-open set of \( Y \) containing \( f(x) \). By (2), \( f^{-1}(V) \) is \( m_X \)-open containing \( x \). Take \( U = f^{-1}(V) \). Then \( f(U) \subseteq V \). This shows that \( f \) is faintly \((m,\mu)\)-continuous.

(2)⇒(3) Let \( F \) be any \( \zeta \)-closed set of \( Y \). Since \( Y - F \) is a \( \zeta \)-open set, by (2), it follows that \( f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F) \) is \( m_X \)-open. Therefore, \( f^{-1}(F) \) is \( m_X \)-closed in \( X \).

(3)⇒(2) Let \( V \) be a \( \zeta \)-open set of \( Y \). Then \( Y - V \) is \( \zeta \)-closed in \( Y \). By (3), \( f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V) \) is \( m_X \)-closed and hence \( f^{-1}(V) \) is \( m_X \)-open in \( X \).

(3)⇒(4) Let \( x \in X \) and \( V \) be a \( \zeta \)-open set of \( Y \) containing \( f(x) \). By (3), it follows that \( X - f^{-1}(V) = f^{-1}(Y - V) = m_X \text{Cl}(f^{-1}(Y - V)) = X - m_X \text{Int}(f^{-1}(V)) \). Therefore, \( f^{-1}(V) = m_X \text{Int}(f^{-1}(V)) \), there exists \( U \in m_X \) containing \( x \) such that \( x \in U \subseteq f^{-1}(V) \). Hence, \( f(U) \subseteq V \). This shows that \( f : (X,m_X) \to (Y,\mu) \) is \((m,\mu)\)-continuous.

(4)⇒(5) Let \( A \) be any subset of \( Y \). Since \( c_\zeta(A) \) is a \( \zeta \)-closed set in \( Y \), by Theorem 3.8(4), \( m_X \text{Cl}(f^{-1}(c_\zeta(A))) = f^{-1}(c_\zeta(A)) \). Thus \( m_X \text{Cl}(f^{-1}(A)) \subseteq m_X \text{Cl}(f^{-1}(c_\zeta(A))) = f^{-1}(c_\zeta(A)) \).

(5)⇒(6) Let \( B \) be any subset of \( Y \). By (5), we have \( X - m_X \text{Int}(f^{-1}(B)) = m_X \text{Cl}(X - f^{-1}(B)) = m_X \text{Cl}(f^{-1}(Y - B)) \subseteq f^{-1}(c_\zeta(Y - B)) = f^{-1}(Y - (i_\zeta(B))) = X - f^{-1}(i_\zeta(B)) \). Hence, \( f^{-1}(i_\zeta(B)) \subseteq m_X \text{Int}(f^{-1}(B)) \).

(6)⇒(3) Let \( F \) be any \( \zeta \)-closed subset of \( Y \). Then \( Y - F = i_\zeta(Y - F) \) because \( Y - F \) is \( \zeta \)-open. By (6), we have \( X - f^{-1}(F) = f^{-1}(Y - F) \subseteq m_X \text{Int}(f^{-1}(Y - F)) = m_X \text{Int}(X - f^{-1}(F)) = X - m_X \text{Cl}(f^{-1}(F)) \). Therefore, \( m_X \text{Cl}(f^{-1}(F)) \subseteq f^{-1}(F) \). Since \( f^{-1}(F) \subseteq m_X \text{Cl}(f^{-1}(F)) \), \( m_X \text{Cl}(f^{-1}(F)) = f^{-1}(F) \). Hence, \( f^{-1}(F) \) is \( m_X \)-closed in \( X \). \[ \square \]

**Definition 3.11.** [12] Let \((X,m_X)\) be an \( m \)-space and \((Y,\mu)\) be a generalized topological space. A function \( f : (X,m_X) \to (Y,\mu) \) is said to be weakly \((m,\mu)\)-continuous at \( x \in X \) if for each \( \mu \)-open set \( V \) containing \( f(x) \), there exists \( U \in m_X \) containing \( x \) such that \( f(U) \subseteq c_\mu(V) \). The function \( f \) is said to be weakly \((m,\mu)\)-continuous if it is weakly \((m,\mu)\)-continuous at each point \( x \in X \).

**Theorem 3.12.** If a function \( f : (X,m_X) \to (Y,\mu) \) is weakly \((m,\mu)\)-continuous, then it is faintly \((m,\mu)\)-continuous.

**Proof.** Let \( x \in X \) and \( V \) be a \( \zeta \)-open set containing \( f(x) \). By Lemma 3.2, there exists a \( \mu \)-open set \( W \) such that \( f(x) \in W \subseteq c_\mu(W) \subseteq V \). Since
$f$ is weakly $(m, \mu)$-continuous, there exists $U \in m_X$ containing $x$ such that $f(U) \subseteq c_\mu(W) \subseteq V$. Therefore, $f$ is faintly $(m, \mu)$-continuous.

**Definition 3.13.** [12] Let $(X, m_X)$ be an $m$-space and $(Y, \mu)$ be a generalized topological space. A function $f : (X, m_X) \to (Y, \mu)$ is said to be almost $(m, \mu)$-continuous at a point $x \in X$ if for each $\mu$-open set $V$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subseteq i_\mu(c_\mu(V))$. The function $f$ is said to be almost-$(m, \mu)$-continuous if it is $(m, \mu)$-continuous at each point $x \in X$.

**Definition 3.14.** A generalized topological space $(X, \mu)$ is said to be $\mu$-almost regular if for each $\mu_r$-closed set $F$ of $X$ and each point $x \notin F$, there exist disjoint $\mu$-open sets $U$ and $V$ of $X$ such that $x \in U$ and $F \subseteq V$.

**Lemma 3.15.** Let $(Y, \mu)$ be a $\mu$-almost regular space. Then, every $\mu_r$-open set is $\zeta$-open.

**Theorem 3.16.** If a function $f : (X, m_X) \to (Y, \mu)$ is faintly $(m, \mu)$-continuous and $(Y, \mu)$ is $\mu$-almost regular, then $f$ is almost $(m, \mu)$-continuous.

**Proof.** Let $x \in X$ and $V$ be any $\mu$-open set of $Y$ containing $f(x)$. Since $i_\mu(c_\mu(V))$ is $\mu_r$-open, by Lemma 3.15, $i_\mu(c_\mu(V))$ is $\zeta$-open. Since $f$ is faintly $(m, \mu)$-continuous, there exists $U \in m_X$ containing $x$ such that $f(U) \subseteq i_\mu(c_\mu(V))$. This shows that $f$ is almost $(m, \mu)$-continuous. 

**Corollary 3.17.** Let $(Y, \mu)$ be a $\mu$-almost regular space. Then for a function $f : (X, m_X) \to (Y, \mu)$, the following properties are equivalent:

1. $f$ is almost $(m, \mu)$-continuous;
2. $f$ is weakly $(m, \mu)$-continuous;
3. $f$ is faintly $(m, \mu)$-continuous.

**Proof.** The proof follows from Theorem 3.12 and 3.16.

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References


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