Some Relationships between the Tangent Polynomials and Bernstein Polynomials

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Abstract

In this paper, we give some interesting identities on the tangent polynomials and Bernstein polynomials.

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1 Introduction

Throughout this paper, let $p$ be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+=\mathbb{N} \cup \{0\}$. As well known definition, the $p$-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^{-t}t$ with $(t,p) = (s,p) = (t,s) = 1$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$, the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x,$$

where $[1, 2, 3, 4]$.
For $n \in \mathbb{N}$, let $g_n(x) = g(x + n)$ be translation. As well known equation, by (1.1), we have

$$\int_{\mathbb{Z}_p} g(x + n)d\mu_1(x) = (-1)^n \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + 2 \sum_{l=0}^{n-1}(-1)^{n-1-l}g(l). \quad (1.2)$$

The tangent polynomials are defined by the generating function as follows:

$$\left(\frac{2}{e^{2t} + 1}\right)e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.3)$$

In the special case, $x = 0$, $T_n(0) = T_n$ are called the $n$-th tangent numbers (see [4]). From (1.3), we note that

$$T_n(x) = \sum_{l=0}^{n} \binom{n}{l} T_l x^{n-l}. \quad (1.4)$$

From (1.2) and (1.3), for $n = 1$, we have

$$\int_{\mathbb{Z}_p} e^{(x+2y)t} d\mu_1(y) = \left(\frac{2}{e^{2t} + 1}\right)e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.5)$$

By (1.5), we obtain

$$\int_{\mathbb{Z}_p} (x + 2y)^n d\mu_1(y) = T_n(x), \text{ for } n \in \mathbb{Z}_+. \quad (1.6)$$

In [1], Kim introduced $p$-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in \mathbb{Z}_p \text{ and } n, k \in \mathbb{Z}_+. \quad (1.6)$$

In this paper, we investigate some properties for the tangent numbers and polynomials. By using these properties, we give some interesting identities on the tangent polynomials and Bernstein polynomials.

### 2 Some identities on the Bernstein and tangent polynomials

From (1.2), we can derive the following recurrence formula for the tangent numbers:

$$T_0 = 1, \text{ and } (T + 2)^n + T_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad (2.1)$$
with usual convention about replacing $T^n$ by $T_n$.

By (1.3), we easily get
\[
\sum_{n=0}^{\infty} T_n(2-x)(-1)^n \frac{t^n}{n!} = \left( \frac{2}{e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}.
\] (2.2)

By (2.2), we obtain the following theorem.

**Theorem 2.1** For $n \in \mathbb{Z}_+$ and $w \in T_p$, we have

\[
T_n(x) = (-1)^n T_n(2-x).
\]

From (1.6), we note that
\[
\int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x) = T_n, \text{ for } n \in \mathbb{Z}_+.
\] (2.3)

By (2.1), for $n \in \mathbb{N}$, we get
\[
T_n(4) - 2^{n+1} = (T + 2 + 2)^n - 2^{n+1}
= \sum_{l=0}^{n} \binom{n}{l} 2^{n-l}(T + 2)^l - 2^{n+1}
= -\sum_{l=0}^{n} \binom{n}{l} 2^{n-l}T_l
= T_n.
\] (2.4)

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.2** For $n \in \mathbb{N}$, we have

\[
T_n(4) = 2^{n+1} + T_n.
\]

By (2.3) and Theorem 2.1, we have the following corollary.

**Corollary 2.3** For $n \in \mathbb{N}$, we have

\[
\int_{\mathbb{Z}_p} (2x + 4)^n d\mu_{-1}(x) = 2^{n+1} + T_n.
\]
By (2.3), Theorem 2.1, and Corollary 2.3, we know that
\[ \int_{\mathbb{Z}_p} (2 - 2x)^n d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} (2x - 2)^n d\mu_{-1}(x) \]
\[ = (-1)^n T_n(-2) \]
\[ = T_n(4) \]
\[ = \int_{\mathbb{Z}_p} (2x + 4)^n d\mu_{-1}(x) \]
\[ = 2^{n+1} + T_n \]
\[ = 2^{n+1} + \int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x). \]

Therefore, we have the following theorem.

**Theorem 2.4** For \( n \in \mathbb{N} \), we have
\[ \int_{\mathbb{Z}_p} (2 - 2x)^n d\mu_{-1}(x) = 2^{n+1} + \int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x). \]

In (1.7), we take the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for one Bernstein polynomials as follows:
\[ \int_{\mathbb{Z}_p} 2^n B_{k,n}(x) d\mu_{-1}(x) \]
\[ = \left( \begin{array}{c} n \\vspace{1em} \\vspace{1em} \\vspace{1em} \vspace{1em} k \end{array} \right) \sum_{l=0}^{n-k} \left( \begin{array}{c} n-k \\vspace{1em} \\vspace{1em} \\vspace{1em} \\vspace{1em} l \end{array} \right) (-1)^{n-k-l}2^l \int_{\mathbb{Z}_p} (2x)^{n-l} d\mu_{-1}(x) \]
\[ = \left( \begin{array}{c} n \\vspace{1em} \\vspace{1em} \\vspace{1em} \vspace{1em} k \end{array} \right) \sum_{l=0}^{n-k} \left( \begin{array}{c} n-k \\vspace{1em} \\vspace{1em} \\vspace{1em} \\vspace{1em} l \end{array} \right) (-1)^{n-k-l}2^l T_{n-l}, \text{ where } n, k \in \mathbb{Z}_+. \]

From the reflection symmetric properties of Bernstein polynomials, we note that
\[ B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n, k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \]

(2.6)

For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), we have
\[ \int_{\mathbb{Z}_p} 2^n B_{k,n}(x) d\mu_{-1}(x) \]
\[ = \left( \begin{array}{c} n \\vspace{1em} \\vspace{1em} \\vspace{1em} \vspace{1em} k \end{array} \right) \sum_{l=0}^{k} \left( \begin{array}{c} k \\vspace{1em} \\vspace{1em} \\vspace{1em} \\vspace{1em} l \end{array} \right) (-1)^{k-l}2^l \int_{\mathbb{Z}_p} (2-2x)^{n-l} d\mu_{-1}(x) \]
\[ = \left( \begin{array}{c} n \\vspace{1em} \\vspace{1em} \\vspace{1em} \vspace{1em} k \end{array} \right) \sum_{l=0}^{k} \left( \begin{array}{c} k \\vspace{1em} \\vspace{1em} \\vspace{1em} \\vspace{1em} l \end{array} \right) (-1)^{k-l}2^l \left( 2^{n-l+1} + \int_{\mathbb{Z}_p} (2x)^{n-l} d\mu_{-1}(x) \right) \]

Therefore, we have the following theorem.
Therefore, we obtain the following theorem.

**Theorem 2.5** For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), we have

\[
\int_{\mathbb{Z}_p} 2^n B_{k,n}(x) d\mu_{-1}(x) = \left( \frac{n}{k} \right) \sum_{l=0}^{k} \left( \frac{k}{l} \right) (-1)^{k-l} 2^l \left( 2^{n-l+1} + T_{n-l} \right).
\]

By (2.5) and Theorem 2.5, we have the following theorem.

**Theorem 2.6** Let \( n, k \in \mathbb{Z}_+ \) with \( n > k \). Then we have

\[
\sum_{l=0}^{n-k} \left( \frac{n-k}{l} \right) (-1)^{n-k-l} 2^l T_{n-l} = \sum_{l=0}^{k} \left( \frac{k}{l} \right) (-1)^{k-l} 2^l \left( 2^{n-l+1} + T_{n-l} \right).
\]

Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then we get

\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{2} \left( \frac{n_i}{k} \right) \right) \sum_{l=0}^{2k} \left( \frac{2k}{l} \right) (-1)^{2k-l} 2^l \int_{\mathbb{Z}_p} (2 - 2x)^{n_1+n_2-l} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{2} \left( \frac{n_i}{k} \right) \right) \sum_{l=0}^{2k} \left( \frac{2k}{l} \right) (-1)^{2k-l} 2^l \left( 2^{n_1+n_2-l+1} + \int_{\mathbb{Z}_p} (2x)^{n_1+n_2-l} d\mu_{-1}(x) \right).
\]

Therefore, we obtain the following theorem.

**Theorem 2.7** For \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \), we have

\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{2} \left( \frac{n_i}{k} \right) \right) \sum_{l=0}^{n_1+n_2-2k} \left( \frac{n_1+n_2-2k}{l} \right) (-1)^{n_1+n_2-2k-l} 2^l \int_{\mathbb{Z}_p} (2x)^{n_1+n_2-l} d\mu_{-1}(x)
\]

\[
= \left( \prod_{i=1}^{2} \left( \frac{n_i}{k} \right) \right) \sum_{l=0}^{n_1+n_2-2k} \left( \frac{n_1+n_2-2k}{l} \right) (-1)^{n_1+n_2-2k-l} 2^l T_{n_1+n_2-l}.
\]

Therefore, by (2.7) and Theorem 2.7, we obtain the following theorem.
Theorem 2.8 Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then we have
\[
\sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k-l} 2^l \left( 2^{n_1+n_2-l+1} + T_{n_1+n_2-l} \right)
\]
\[= \sum_{l=0}^{n_1+n_2-2k} \binom{n_1 + n_2 - 2k}{l} (-1)^{n_1+n_2-2k-l} 2^l T_{n_1+n_2-l}.
\]

For \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( N_3 = n_1 + n_2 + n_3 > 3k \), by the symmetry of Bernstein polynomials, we see that
\[
\int_{\mathbb{Z}_p} 2^{N_3} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) \, d\mu_{-1}(x)
\]
\[= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{n_1+n_2+n_3-l+1} + T_{n_1+n_2+n_3-l} \right).
\]

Therefore, we have the following theorem.

Theorem 2.9 For \( n_1, n_2, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + n_3 > 3k \), we have
\[
\int_{\mathbb{Z}_p} 2^{n_1+n_2+n_3} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) \, d\mu_{-1}(x)
\]
\[= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{n_1+n_2+n_3-l+1} + T_{n_1+n_2+n_3-l} \right).
\]

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:
\[
\int_{\mathbb{Z}_p} 2^{N_3} B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) \, d\mu_{-1}(x)
\]
\[= \left( \prod_{i=1}^{3} \binom{n_i}{k} \right) \sum_{l=0}^{N_3-3k} (-1)^{N_3-3k-l} 2^l \binom{N_3-3k}{l} T_{N_3-l},
\]
where \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( N_3 = n_1 + n_2 + n_3 > 3k \).

Therefore, by Theorem 2.9, we obtain the following theorem.

Theorem 2.10 Let \( n_1, n_2, n_3, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 + n_3 > 3k \). Then we have
\[
\sum_{l=0}^{3k} \binom{3k}{l} (-1)^{3k-l} 2^l \left( 2^{n_1+n_2+n_3-l+1} + T_{n_1+n_2+n_3-l} \right)
\]
\[= \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^{n_1+n_2+n_3-3k-l} 2^l \binom{n_1 + n_2 + n_3 - 3k}{l} T_{n_1+n_2+n_3-l}.
\]
Using the above theorem and mathematical induction, we obtain the following theorem.

**Theorem 2.11** Let \( m \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+ \) with \( N_m = n_1 + \cdots + n_m > mk \), the multiplication of the sequence of Bernstein polynomials \( B_{k,n_1}(x), \ldots, B_{k,n_m}(x) \) with different degrees under fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) can be given as

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{m} 2^{n_i} B_{k,n_i}(x) \right) d\mu(x)
= \left( \prod_{i=1}^{m} \binom{n_i}{k} \right) \sum_{l=0}^{mk} \binom{mk}{l} (-1)^{mk-l} 2^l (2^{N_m-l+1} T_{N_m-l}).
\]

We also easily see that

\[
\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{m} 2^{n_i} B_{k,n_i}(x) \right) d\mu(x)
= \left( \prod_{i=1}^{m} \binom{n_i}{k} \right) \sum_{l=0}^{N_m-mk} \binom{N_m-mk}{l} (-1)^{N_m-mk-l} 2^l T_{N_m-l}.
\tag{2.8}
\]

By Theorem 2.11 and (2.8), we have the following corollary.

**Corollary 2.12** Let \( m \in \mathbb{N} \). For \( n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+ \) with \( n_1 + \cdots + n_m > mk \), we have

\[
\sum_{l=0}^{mk} \binom{mk}{l} (-1)^{mk-l} 2^l (2^{n_1+\cdots+n_m-l+1} + T_{n_1+\cdots+n_m-l})
= \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1+\cdots+n_m-mk}{l} (-1)^{n_1+\cdots+n_m-mk-l} 2^l T_{n_1+\cdots+n_m-l}.
\]

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**References**


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