On Intra-Regular Ternary Semihypergroups

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Abstract

In this paper we give some characterizations of the intra-regular ternary semihypergroups in terms of bi-hyperideals and quasi-hyperideals, bi-hyperideals and left hyperideals, bi-hyperideals and right hyperideals of ternary semihypergroups.

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1 Introduction

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty [3]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. In this paper we give some characterizations of the intra-regular ternary semihypergroups in terms of bi-hyperideals and quasi-hyperideals, bi-hyperideals and left hyperideals, bi-hyperideals and right hyperideals of ternary semihypergroups.

2 Preliminary Notes

Definition 2.1 A map $\circ : H \times H \times H \to \mathcal{P}^*(H)$ is called ternary hyper-operation on the set $H$, where $H$ is a nonempty set and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subset of $H$. 
Definition 2.2 A ternary hyperstructure is called the pair $(H, \circ)$, where \( \circ \) is a ternary hyperoperation on the set \( H \).

Definition 2.3 A hyperstructure $(H, \circ)$ is called a ternary semihypergroup if for all \( x, y, z, a, b \in H \)
\[(x \circ a \circ y) \circ b \circ z = x \circ (a \circ y \circ b) \circ z = x \circ a \circ (y \circ b \circ z),\]
which means that
\[
\bigcup_{u \in x \circ a \circ y} u \circ b \circ z = \bigcup_{v \in a \circ y \circ b} x \circ v \circ z = \bigcup_{w \in y \circ b \circ z} x \circ a \circ w.
\]

If \( x \in H \) and \( A, B, C \) are nonempty subsets of \( H \), then
\[A \circ B \circ C = \bigcup_{a \in A, b \in B, c \in C} a \circ b \circ c, \quad A \circ H \circ x = A \circ H \circ \{x\}, \quad x \circ H \circ B = \{x\} \circ H \circ B.\]

Definition 2.4 A nonempty subset \( B \) of a ternary semihypergroup \( H \) is called a ternary subsemihypergroup of \( H \) if \( B \circ B \circ B \subseteq B \).

Definition 2.5 Let \( H \) be a ternary semihypergroup. \( \emptyset \neq A \subseteq H \), \( A \) is called a right (resp. left) hyperideal of \( H \) if \( A \circ H \circ H \subseteq A \) (resp. \( H \circ H \circ A \subseteq A \) ).

Definition 2.6 Let \( H \) be a ternary semihypergroup. \( \emptyset \neq A \subseteq H \), \( A \) is called a lateral hyperideal of \( H \) if \( H \circ A \circ H \subseteq A \).

If \( A \) is a right, left and lateral hyperideal of \( H \), then it is called an hyperideal of \( H \).

Definition 2.7 Let \( H \) be a ternary semihypergroup. \( \emptyset \neq B \subseteq H \), \( B \) is called a bi-hyperideal of \( H \) if \( B \circ H \circ B \subseteq B \).

Definition 2.8 Let \( H \) be a ternary semihypergroup. \( \emptyset \neq Q \subseteq H \), \( Q \) is called a quasi-hyperideal of \( H \) if \( (Q \circ H \circ H) \cap (H \circ H \circ Q) \subseteq Q \).

Let \( H \) be a ternary semihypergroup and \( a \in H \). The right (resp. left) hyperideal of \( H \) generated by \( a \), denoted by \( R(a) \) (resp. \( L(a) \)) is of the form:
\[R(a) = a \cup a \circ H \circ H \text{ and } L(a) = a \cup H \circ H \circ a.\]

The ideal of \( H \) generated by \( a \), denoted by \( I(a) \), is the form
\[I(a) = a \cup H \circ H \circ a \cup a \circ H \circ H \cup H \circ H \circ a \circ H \circ H.\]

The quasi- (resp. bi-) hyperideal of \( H \) generated by \( a \), denoted by \( Q(a) \) (resp. \( B(a) \)), is of the form
\[Q(a) = a \cup (a \circ H \circ H \cap H \circ H \circ a) \text{ and } B(a) = a \cup a \circ H \circ a \cup a \circ H \circ H \circ H \circ a.\]
3 Main Results

Now we give some characterizations of the intra-regular ternary semihypergroups in terms of bi-hyperideals, quasi-hyperideals, left hyperideals and right hyperideals of ternary semihypergroups.

**Definition 3.1** A ternary semihypergroup $H$ is said to be intra-regular if

$$a \in H \circ H \circ a \circ H \circ a \circ H \circ H$$

for all $a \in H$.

**Theorem 3.2** Let $H$ be a ternary semihypergroup. Then:

1. $H$ is intra-regular if and only if for a bi-hyperideal $B$ and a quasi-hyperideal $Q$ of $H$, we have $B \cap Q \subseteq H \circ H \circ B \circ H \circ Q \circ H \circ H$.

2. $H$ is intra-regular if and only if for a bi-hyperideal $B$ and a quasi-hyperideal $Q$ of $H$, we have $B \cap Q \subseteq H \circ H \circ Q \circ H \circ B \circ H \circ H$.

**Proof.**

(1) Assume that $H$ is intra-regular. Let $B$ be a bi-hyperideal of $H$ and $Q$ a quasi-hyperideal of $H$. Let $a \in B \cap Q$. By assumption, $a \in H \circ H \circ a \circ H \circ a \circ H \circ H$. Then $a \in H \circ H \circ a \circ H \circ a \circ H \circ H \subseteq H \circ H \circ a \circ H \circ (H \circ H \circ a \circ H \circ a \circ H \circ H \circ a \circ H \circ H \circ H \circ H \circ H \circ H \circ H \subseteq H \circ H \circ B \circ H \circ Q \circ H \circ H$.

Hence $B \cap Q \subseteq H \circ H \circ B \circ H \circ Q \circ H \circ H$.

Conversely, assume that for a bi-hyperideal $B$ and a quasi-hyperideal $Q$ of $H$, we have $B \cap Q \subseteq H \circ H \circ B \circ H \circ Q \circ H \circ H$. Let $a \in H$. Consider:

$$a \in B(a) \cap Q(a)$$

$$\subseteq H \circ H \circ B(a) \circ H \circ Q(a) \circ H \circ H$$

$$= H \circ H \circ (a \cup a \circ H \circ a \cup a \circ H \circ H \circ H \circ a) \circ H \circ (a \cup (a \circ H \circ H \circ H \circ a) \circ a \circ H \circ H \circ a) \circ H \circ H$$

$$\subseteq (H \circ H \circ a \cup H \circ a \cup H \circ a \circ H \circ H \circ H \circ a \circ H \circ H \circ a \circ H) \circ H \circ H$$

$$\subseteq H \circ H \circ a \circ H \circ a \circ H \circ H \cup a \circ H \circ H \circ a \circ H \circ H \circ H \circ H$$

$$\subseteq H \circ H \circ a \circ H \circ a \circ H \circ H$$

This proves that $H$ is intra-regular.

(2) Assume that $H$ is intra-regular. Let $B$ be a bi-hyperideal of $H$ and $Q$ a quasi-hyperideal of $H$. Let $a \in B \cap Q$. By assumption, $a \in H \circ H \circ a \circ H \circ a \circ H \circ H \circ H \subseteq H \circ H \circ (H \circ H \circ a \circ H \circ a \circ H \circ H \circ a \circ H \circ H \circ a \circ H \circ a \circ H \circ H \circ H \circ H \circ H \circ H \subseteq H \circ H \circ Q \circ H \circ (B \circ H \circ H \circ B \circ H \circ H \circ H \circ H \subseteq H \circ H \circ Q \circ H \circ B \circ H \circ H \circ H \circ H$.

Hence $B \cap Q \subseteq H \circ H \circ Q \circ H \circ B \circ H \circ H$. 


Conversely, assume that for a bi-hyperideal $B$ and a quasi-hyperideal $Q$ of $H$, we have $B \cap Q \subseteq H \circ H \circ Q \circ H \circ B \circ H \circ H$. Let $a \in H$. Consider
\[
\begin{align*}
a & \in B(a) \cap Q(a) \\
& \subseteq H \circ H \circ Q(a) \circ H \circ B(a) \circ H \circ H \\
& = H \circ H \circ (a \cup (a \circ H \circ H \cap H \cup H \circ a)) \circ H \circ (a \cup a \circ H \circ a \cup a \circ H \circ a) \circ H \\
& \subseteq H \circ H \circ (a \cup H \circ H \cup H \circ a \circ a \circ H \circ H \cup a \circ H \circ H \circ a \circ H \circ a) \circ H \\
& \subseteq (H \circ H \circ a \cup H \circ H \circ a \circ H \circ H \circ a \circ H \circ a) \circ H \circ (a \circ H \circ H \cup a \circ H \circ a \circ H \circ H \cup a \circ H \circ H \cup H \circ a \circ H \circ a \circ H \circ H) \\
& \subseteq (H \circ H \circ a \cup H \circ H \circ a \circ H \circ a) \circ H \circ (a \circ H \circ H) \\
& \subseteq H \circ H \circ a \circ H \circ a \circ H \circ H \circ H \circ a \circ H \circ a \circ H \circ H \\
& \subseteq H \circ H \circ a \circ H \circ a \circ H \circ H.
\end{align*}
\]
Therefore $H$ is intra-regular.

**Theorem 3.3** Let $H$ be a ternary semihypergroup. Then:

1. $H$ is intra-regular if and only if for a left hyperideal $L$ and a bi-hyperideal $B$ of $H$, we have $L \cap B \subseteq L \circ H \circ B \circ H \circ H$.

2. $H$ is intra-regular if and only if for a right hyperideal $R$ and a bi-hyperideal $B$ of $H$, we have $B \cap R \subseteq H \circ H \circ B \circ H \circ R$.

**Proof.** (1) Assume that $H$ is intra-regular. Let $L$ be a left hyperideal of $H$ and $B$ a bi-hyperideal of $H$. Let $a \in L \cap B$. By assumption, $a \in H \circ a \circ B \circ H \circ H$. Since $a \in H \circ H \circ a \circ H \circ a \circ H \circ H \subseteq H \circ H \circ L \circ H \circ B \circ H \circ H \subseteq L \circ H \circ B \circ H \circ H$, we have $a \in L \circ H \circ B \circ H \circ H$.

Conversely, for a left hyperideal $L$ and a bi-hyperideal $B$ of $H$, we have $L \cap B \subseteq L \circ H \circ B \circ H \circ H$.

For $a \in H$, we have
\[
\begin{align*}
a & \in L(a) \cap B(a) \\
& \subseteq L(a) \circ H \circ B(a) \circ H \circ H \\
& = (a \cup H \circ H \circ a) \circ H \circ (a \cup a \circ H \circ a \cup a \circ H \circ H \circ a \circ H \circ a) \circ H \circ H \\
& \subseteq (a \cup H \circ H \circ a) \circ H \circ (a \circ H \circ H \cup a \circ H \circ H \cup a \circ H \circ H \cup a \circ H \circ H \cup a \circ H \circ H) \\
& \subseteq (a \cup H \circ H \circ a) \circ H \circ (a \circ H \circ H) \\
& \subseteq (a \circ H \circ a \circ H \circ H \cup H \circ a \circ H \circ a \circ H \circ H). \quad \text{(if $a \in a \circ H \circ a \circ H \circ H$, then $a \in a \circ H \circ a \circ H \circ H \subseteq a \circ H \circ a \circ H \circ a \circ H \circ a \circ H \circ H \subseteq H \circ H \circ a \circ H \circ a \circ H \circ H$. If $a \in H \circ H \circ a \circ H \circ a \circ H \circ H$, it is obvious.)
\]
Therefore $H$ is intra-regular.

(2) Assume that $H$ is intra-regular. Let $R$ be a right hyperideal of $H$ and $B$ a bi-hyperideal of $H$. Let $a \in R \cap B$. By assumption, $a \in H \circ H \circ a \circ H \circ a \circ H \circ H$. Since $a \in H \circ H \circ a \circ H \circ a \circ H \circ H \subseteq H \circ H \circ B \circ H \circ R \circ H \circ H \subseteq H \circ H \circ B \circ H \circ R$, we have $B \cap R \subseteq H \circ H \circ B \circ H \circ R$. 

Conversely, for a right hyperideal \( R \) and a bi-hyperideal \( B \) of \( H \), we have 
\[ B \cap R \subseteq H \circ H \circ B \circ H \circ R. \]

For \( a \in H \), we have 
\[ a \in B(a) \cap R(a) \subseteq H \circ H \circ B(a) \circ H \circ R(a) \]

\[ = H \circ H \circ (a \cup a \circ H \circ a \cup a \circ H \circ H \circ a) \circ H \circ (a \cup a \circ H \circ H) \]

\[ \subseteq (H \circ H \circ a \cup H \circ a \circ H \circ a \cup H \circ a \circ H \circ H \circ a) \circ H \circ H \circ a \circ H \circ H \]

\[ \subseteq (H \circ H \circ a \circ H \circ a) \cup (H \circ H \circ a \circ H \circ a \circ H \circ H). \]

If \( a \in H \circ H \circ a \circ H \circ a \circ a \circ a \circ a \circ a \circ H \circ a \subseteq H \circ H \circ a \circ H \circ a \circ H \circ a \circ H \circ H \circ a \subseteq H \circ H \circ a \circ H \circ a \circ H \circ a \circ H \circ a \), then \( a \in H \circ H \circ a \circ H \circ a \circ H \circ a \circ H \), it is obvious. Therefore \( a \in H \circ H \circ a \circ H \circ a \circ H \circ H \) for any cases. Hence \( H \) is intra-regular.

Using Theorem 3.2 and Theorem 3.3, we obtain:

**Theorem 3.4** Let \( H \) be a ternary semihypergroup and \( a \in H \). Then the following are equivalent:

1. \( H \) is intra-regular.
2. \( B(a) \cap Q(a) \subseteq H \circ H \circ B(a) \circ H \circ Q(a) \circ H \circ H. \)
3. \( B(a) \cap Q(a) \subseteq H \circ H \circ Q(a) \circ H \circ B(a) \circ H \circ H. \)
4. \( L(a) \cap B(a) \subseteq L(a) \circ H \circ B(a) \circ H \circ H. \)
5. \( B(a) \cap R(a) \subseteq H \circ H \circ H \circ H \circ B(a) \circ H \circ R(a). \)

**References**


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