Approximate Derivations on Banach Algebras
Associated with Jensen Functional Equation

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Abstract. In this paper, we examine the generalized Hyers–Ulam stability of derivations associated with the Jensen functional equation

$$f\left(\frac{x-y}{n} + z\right) + f\left(\frac{y-z}{n} + x\right) + f\left(\frac{z-x}{n} + y\right) = f(x) + f(y) + f(z)$$

on Banach algebras for any fixed nonzero integer $n$.

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1. Introduction and Preliminaries

The stability problem for ring derivations on Banach algebras was considered by Miura et al. in [8]: under suitable conditions every approximate ring derivation on Banach algebra is an exact ring derivation. Šemrl [19] obtained the first stability result concerning derivations between operator algebras. The study of stability problems as just mentioned originated from a famous talk given by Ulam [20] in 1940: under what condition does there exist a homomorphism near an approximate homomorphism? In following year, Hyers [6] was answered affirmatively the question of Ulam and the result can be formulated as follows: if \( \varepsilon \geq 0 \) and \( f : E_1 \to E_2 \) is a mapping with \( E_1, E_2 \) Banach spaces such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon
\]

for all \( x, y \in E_1 \), then there exists a unique additive mapping \( h : E_1 \to E_2 \) such that

\[
\|f(x) - h(x)\| \leq \varepsilon.
\]

Moreover, if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each \( x \in E_1 \), then \( h \) is \( \mathbb{R} \)-linear.

The method which was provided by Hyers, and which produces the additive mapping \( h \), is called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations.

The theorem of Hyers was generalized by Aoki [1] and Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [14] independently introduced the unbounded Cauchy difference and was the first to verify the stability of the linear mapping between Banach spaces: if there exist \( \varepsilon > 0 \) and \( 0 \leq p < 1 \) such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \quad \forall x, y \in E_1,
\]

then there exists a unique \( \mathbb{R} \)-linear mapping \( T : E_1 \to E_2 \) such that

\[
\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p
\]

for all \( x \in E_1 \). A generalized version of the theorem of Rassias was given by Găvruta and Jung [5, 7]. In 1991, Gajda [4] answered the question for the case \( p > 1 \), which was raised by Rassias [15]. It was shown by Gajda [4], as well as by Rassias and Šemrl [16], that the Rassias’ type stability result is not valid for \( p = 1 \). In particular, J.M. Rassias [17] generalized the Hyers’ stability result
by presenting a weaker condition controlled by a product of different powers of norms.

On the other hand, Moslehian and Najati [9] introduced the Hyers–Ulam stability of an additive functional inequality
\[ \|f\left(\frac{x - y}{2} + z\right) + f\left(\frac{y - z}{2} + x\right) + f\left(\frac{z - x}{2} + y\right)\| \leq \|f(x + y + z)\| \] (1.1)
and then have investigated the general solution and the Hyers-Ulam stability problem for the functional inequality. Many interesting results of the stability problems to a number of functional equations and functional inequalities involving derivations have been investigated.

In this paper, we take account of a modified and general Jensen functional equation
\[ f\left(\frac{x - y}{n} + z\right) + f\left(\frac{y - z}{n} + x\right) + f\left(\frac{z - x}{n} + y\right) = f(x) + f(y) + f(z) \] (1.2)
for any fixed nonzero integer \( n \). First of all, it is easy to see that a function \( f \) satisfies the equation (1.2) if and only if \( f(x) - f(0) \) is additive. In the sequel, we investigate the generalized Hyers–Ulam stability of derivations associated with the equation (1.2) on Banach algebras.

2. Generalized Hyers-Ulam stability via direct method

Before taking up the main subject, given a mapping \( f : X \to X \), we define the difference operator \( Df : X^3 \to X \) by
\[
D_1f(\lambda x, y, z) := f\left(\frac{\lambda x - y}{n} + z\right) + f\left(\frac{y - z}{n} + \lambda x\right) + f\left(\frac{z - \lambda x}{n} + y\right) - \lambda f(x) - f(y) - f(z),
\]
\[
D_2f(a, b) := f(ab) - af(b) - f(a)b
\]
for all \( x, y, z, a, b \in X \) and all \( \lambda \in T_1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and for any fixed nonzero integer \( n \), where \( D_1f \) acts as a difference operator of the Jensen equation (1.2) and \( D_2f \) acts as a difference operator of derivation equation \( h(ab) = ah(b) + h(a)b \). Throughout the paper, we assume that \( X \) is a Banach algebra.

Theorem 2.1. Suppose that a mapping \( f : X \to X \) with \( f(0) = 0 \) satisfies the functional inequality
\[ \|D_1f(\lambda x, y, z) - D_2f(a, b)\| \leq \varphi(x, y, z, a, b) \] (2.1)
and that the function $\varphi : X^5 \to \mathbb{R}^+$ satisfies
\[
\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i y, 2^i z, 0, 0) < \infty, \quad \lim_{k \to \infty} \frac{\varphi(0, 0, 0, 2^k a, 2^k b)}{2^{2k}} = 0 \quad (2.2)
\]
for all $x, y, z, a, b \in X$. Then there exists a unique linear derivation $h : X \to X$ defined by $h(x) = \lim_{k \to \infty} \frac{1}{2^k} f(2^k x)$ such that
\[
\|f(x) - h(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i x, (1-n)2^i x, 0, 0) \quad (2.3)
\]
for all $x \in X$.

**Proof.** Letting $\lambda := 1, y := x$ and $a = b := 0$ in (2.1), we have
\[
\left\| f\left( \frac{x - z}{n} + x \right) + f\left( \frac{z - x}{n} + x \right) - 2f(x) \right\| \leq \varphi(x, x, z, 0, 0) \quad (2.4)
\]
for all $x, z \in X$. Replacing $z$ by $x - nx$ in (2.4), we get
\[
\|f(2x) - 2f(x)\| \leq \varphi(x, x, (1-n)x, 0, 0), \quad (2.5)
\]
that is,
\[
\|f(x) - \frac{1}{2^l} f(2^l x)\| \leq \frac{1}{2^l} \varphi(x, x, (1-n)x, 0, 0) \quad (2.6)
\]
for all $x \in X$. It follows from (2.6) that
\[
\left\| \frac{f(2^l x)}{2^l} - \frac{f(2^m x)}{2^m} \right\| \leq \sum_{i=l}^{m-1} \left\| \frac{1}{2^i} f(2^i x) - \frac{1}{2^{i+1}} f(2^{i+1} x) \right\| \\
= \sum_{i=l}^{m-1} \frac{1}{2^i} \left\| f(2^i x) - \frac{1}{2} f(2^{i+1} x) \right\| \\
\leq \sum_{i=l}^{m-1} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i x, (1-n)2^i x, 0, 0) \quad (2.7)
\]
for all nonnegative integers $m$ and $l$ with $m > l \geq 0$ and $x \in X$. Since the right-hand side of (2.7) tends to zero as $l \to \infty$, by the convergence of the series (2.2), we obtain that the sequence $\{\frac{f(2^m x)}{2^m}\}$ is Cauchy for all $x \in X$. Due to the fact that $X$ is complete, it follows that the sequence $\{\frac{f(2^m x)}{2^m}\}$ converges in $X$. So one can define a mapping $h : X \to X$ as
\[
h(x) = \lim_{m \to \infty} \frac{f(2^m x)}{2^m}, \quad x \in X.
\]
Moreover, letting $l = 0$ and taking $m \to \infty$ in (2.7), we get
\[
\|f(x) - h(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i x, (1-n)2^i x, 0, 0)
\]
for all $x \in X$. 
It follows from (2.1) and (2.2) that
\[ \| h\left(\frac{\lambda x - y}{n} + z\right) + h\left(\frac{y - z}{n} + \lambda x\right) + h\left(\frac{z - \lambda x}{n} + y\right) - \lambda h(x) - h(y) - h(z) \| \]
\[ = \lim_{k \to \infty} \frac{1}{2^k} \| D_1 f(2^k \lambda x, 2^k y, 2^k z) \| \]
\[ \leq \lim_{k \to \infty} \frac{1}{2^k} \varphi(2^k x, 2^k y, 2^k z, 0, 0) = 0 \]
for all \( x, y, z \in X \) and all \( \lambda \in T_1 \). So the mapping \( h \) is additive and \( h(\lambda x) = \lambda h(x) \) for all \( x \in X \) and all \( \lambda \in T_1 \). Then the additive mapping \( h \) reduces to \( \mathbb{C} \)-linear mapping [12, 13]. In addition, we figure out that
\[ \| h(ab) - ah(b) - h(a)b \| = \lim_{k \to \infty} \frac{1}{2^k} \| D_2 f(2^k a, 2^k b) \| \]
\[ \leq \lim_{k \to \infty} \frac{1}{2^k} \varphi(0, 0, 2^k a, 2^k b) = 0 \]
for all \( a, b \in X \). Thus the mapping \( h \) is linear derivation on \( X \).

Next, let \( h' : X \to X \) be another linear derivation satisfying
\[ \| f(x) - h'(x) \| \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i x, (1 - n)2^i x, 0, 0) \]
for all \( x \in X \). Then, we have
\[ \| h(x) - h'(x) \| = \left\| \frac{1}{2^k} h(2^k x) - \frac{1}{2^k} h'(2^k x) \right\| \]
\[ \leq \frac{1}{2^k} (\| h(2^k x) - f(2^k x) \| + \| f(2^k x) - h'(2^k x) \|) \]
\[ \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+k}} \varphi(2^{i+k} x, 2^{i+k} x, (1 - n)2^{i+k} x, 0, 0) \]
\[ = \sum_{i=k}^{\infty} \frac{1}{2^i} \varphi(2^i x, 2^i x, (1 - n)2^i x, 0, 0) \]
for all \( k \in \mathbb{N} \) and all \( x \in X \). Taking the limit as \( k \to \infty \), we conclude that
\[ h(x) = h'(x) \]
for all \( x \in X \). This proves the uniqueness of the linear derivation \( h \) satisfying (2.3).

The previous theorem implies the following results.

**Corollary 2.2.** Let \( r_i > 0 \) for \( i = 1, \ldots, 5 \) with \( \sum_{i=1}^{3} r_i < 1, r_4 + r_5 < 2 \) and \( \theta_1, \theta_2 \geq 0 \). If a mapping \( f : X \to X \) with \( f(0) = 0 \) satisfies the following functional inequality
\[ \| D_1 f(\lambda x, y, z) - D_2 f(a, b) \| \leq \theta_1 \| x \|^{r_1} \| y \|^r_2 \| z \|^{r_3} + \theta_2 \| a \|^{r_4} \| b \|^{r_5} \]
for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that
\[
\|f(x) - h(x)\| \leq \frac{\theta_1 (1 - n)^{r_3}}{2 - 2^r} \|x\|^r
\]
for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.

**Corollary 2.3.** Let $r_i$ be positive reals with $r_i < 1$ for $i = 1, 2, 3$ and $r_4 + r_5 < \frac{r}{2}$, and $\theta_i \geq 0$ for $i = 1, \ldots, 4$. If a mapping $f : X \to X$ with $f(0) = 0$ satisfies the functional inequality
\[
\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} + \theta_4 \|a\|^{r_4} + \theta_5 \|b\|^{r_5}
\]
for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that
\[
\|f(x) - h(x)\| \leq \frac{\theta_1 \|x\|^{r_1}}{2 - 2^{r_1}} + \frac{\theta_2 \|x\|^{r_2}}{2 - 2^{r_2}} + \frac{\theta_3 (1 - n)^{r_3} \|x\|^{r_3}}{2 - 2^{r_3}}
\]
for all $x \in X$.

**Corollary 2.4.** Let $r_i$ be positive reals with $r_i < 1$ for $i = 1, 2, 3$ and $r_4, r_5 < \frac{r}{2}$, and $\theta_i \geq 0$ for $i = 1, \ldots, 5$. If a mapping $f : X \to X$ with $f(0) = 0$ satisfies the functional inequality
\[
\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} + \theta_4 \|a\|^{r_4} + \theta_5 \|b\|^{r_5}
\]
for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that
\[
\|f(x) - h(x)\| \leq \frac{\theta_1 \|x\|^{r_1}}{2 - 2^{r_1}} + \frac{\theta_2 \|x\|^{r_2}}{2 - 2^{r_2}} + \frac{\theta_3 (1 - n)^{r_3} \|x\|^{r_3}}{2 - 2^{r_3}}
\]
for all $x \in X$.

**Theorem 2.5.** Suppose that a mapping $f : X \to X$ satisfies the functional inequality
\[
\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \varphi(x, y, z, a, b)
\]
and that the function $\varphi : X^3 \to \mathbb{R}^+$ satisfies
\[
\sum_{i=0}^{\infty} 2^{i} \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, \frac{z}{2^{i+1}}, 0, 0\right) < \infty, \quad \lim_{k \to \infty} 2^{2k} \varphi\left(0, 0, 0, \frac{a}{2^k}, \frac{b}{2^k}\right) = 0
\]
for all $x, y, z, a, b \in X$. Then there exists a unique linear derivation $h : X \to X$ defined by $h(x) = \lim_{k \to \infty} 2^{k} f\left(\frac{x}{2^k}\right)$ such that
\[
\|f(x) - h(x)\| \leq \sum_{i=0}^{\infty} 2^{i} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{1 - n) x}{2^{i+1}}, 0, 0\right)
\]
for all $x \in X$. 
Proof. We observe that $f(0) = 0$ because $\varphi(0, 0, 0, 0, 0) = 0$ by the condition (2.9). Now, if we replace $x$ by $\frac{x}{2}$ in (2.5), then we have

\[
\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{(1-n)x}{2}, 0, 0\right)
\]

for all $x \in X$. Thus it follows from the last inequality that

\[
\left\| f(x) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{i=0}^{m-1} 2^i \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{(1-n)x}{2^{i+1}}, 0, 0\right)
\]

(2.11)

for all nonnegative integer $m$ and all $x \in X$.

The rest of proof is similar to the corresponding part of Theorem 2.1. \qed

The following corollaries are immediate consequences of Theorem 2.5.

**Corollary 2.6.** Let $r_i > 0$ for $i = 1, \cdots, 5$ with $\sum_{i=1}^{3} r_i > 1, r_4 + r_5 > 2$ and $\theta_1, \theta_2 \geq 0$. If a mapping $f : X \to X$ satisfies the functional inequality

\[
\left\| D_1 f(\lambda x, y, z) - D_2 f(a, b) \right\| \leq \theta_1 \|x\|^{r_1} \|y\|^{r_2} \|z\|^{r_3} + \theta_2 \|a\|^{r_4} \|b\|^{r_5}
\]

for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that

\[
\left\| f(x) - h(x) \right\| \leq \frac{\theta_1(1-n)r_3}{2^{r_1} - 2} \|x\|^{r_1}
\]

for all $x \in X$, where $r = \sum_{i=1}^{3} r_i$.

**Corollary 2.7.** Let $r_i$ be positive reals with $r_i > 1$ for $i = 1, 2, 3$ and $r_4 + r_5 > 2$, and $\theta_i \geq 0$ for $i = 1, \cdots, 4$. If a mapping $f : X \to X$ satisfies the functional inequality

\[
\left\| D_1 f(\lambda x, y, z) - D_2 f(a, b) \right\| \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} + \theta_4 \|a\|^{r_4} \|b\|^{r_5}
\]

for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that

\[
\left\| f(x) - h(x) \right\| \leq \frac{\theta_1 \|x\|^{r_1}}{2^{r_1} - 2} + \frac{\theta_2 \|x\|^{r_2}}{2^{r_2} - 2} + \frac{\theta_3(1-n)r_3\|x\|^{r_3}}{2^{r_3} - 2}
\]

for all $x \in X$.

**Corollary 2.8.** Let $r_i$ be positive reals with $r_i > 1$ for $i = 1, 2, 3$ and $r_4, r_5 > 2$, and $\theta_i \geq 0$ for $i = 1, \cdots, 5$. If a mapping $f : X \to X$ satisfies the functional inequality

\[
\left\| D_1 f(\lambda x, y, z) - D_2 f(a, b) \right\| \leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} + \theta_3 \|z\|^{r_3} + \theta_4 \|a\|^{r_4} + \theta_5 \|b\|^{r_5}
\]
for all $x, y, z, a, b \in X$, then there exists a unique linear derivation $h : X \to X$ such that
\[
\| f(x) - h(x) \| \leq \frac{\theta_1 \| x \|^r_1}{2^{r_1} - 2} + \frac{\theta_2 \| x \|^r_2}{2^{r_2} - 2} + \frac{\theta_3 (1 - n)^r_3 \| x \|^r_3}{2^{r_3} - 2}
\]
for all $x \in X$.

3. ALTERNATIVE GENERALIZED HYERS–ULAM STABILITY

From now on, we investigate the generalized Hyers–Ulam stability of derivations associated with the functional equation (1.2) by using alternative property of perturbing term of the equation (1.2).

**Theorem 3.1.** Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality
\[
\| D_1 f(\lambda x, y, z) - D_2 f(a, b) \| \leq \varphi_1(x, y, z) + \varphi_2(a, b)
\]
and that there exist constants $L_1$ and $L_2$ with $0 < L_1, L_2 < 1$ for which the functions $\varphi_1 : X^3 \to \mathbb{R}^+$ and $\varphi_2 : X^2 \to \mathbb{R}^+$ satisfy the property
\[
\begin{align*}
\varphi_1(2x, 2y, 2z) &\leq 2L_1 \varphi_1(x, y, z), \\
\varphi_2(2a, 2b) &\leq 4L_2 \varphi_2(a, b)
\end{align*}
\]
for all $x, y, z, a, b \in X$. Then there exists a unique linear derivation $h : X \to Y$ given by $h(x) = \lim_{k \to \infty} \frac{1}{2^n} f(2^k x)$ such that
\[
\| f(x) - h(x) \| \leq \frac{\varphi_1(x, x, (1 - n)x)}{2(1 - L_1)} + \varphi_2(0, 0)
\]
for all $x \in X$.

**Proof.** Letting $\lambda := 1, y := x, z := x - nx$ and $a = b := 0$ in (3.1), we get
\[
\| f(2x) - 2f(x) \| \leq \varphi_1(x, x, (1 - n)x) + \varphi_2(0, 0)
\]
that is,
\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} \varphi_1(x, x, (1 - n)x) + \frac{1}{2} \varphi_2(0, 0)
\]
for all $x \in X$. It follows from (3.4) and (3.2) that
\[
\left\| \frac{f(2^m x)}{2^l} - \frac{f(2^l x)}{2^m} \right\| \leq \sum_{i=l}^{m-1} \frac{\varphi_1(2^i x, 2^i x, (1 - n)2^i x) + \varphi_2(0, 0)}{2^{i+1}} \leq \sum_{i=l}^{m-1} \frac{(2L_1)^i \varphi_1(x, x, (1 - n)x) + \varphi_2(0, 0)}{2^{i+1}} \leq \frac{\varphi_1(x, x, (1 - n)x)}{2(1 - L_1)} + \varphi_2(0, 0)
\]
for all nonnegative integers $m$ and $l$ with $m > l \geq 0$ and $x \in X$. Since the sequence $\{\frac{f(2^m x)}{2^m}\}$ is Cauchy for all $x \in X$, we can define a mapping $h : X \rightarrow Y$ by
\[
h(x) = \lim_{m \rightarrow \infty} \frac{f(2^m x)}{2^m}, \quad x \in X.
\]
Moreover, letting $l = 0$ and $m \rightarrow \infty$ in the last inequality yields
\[
\|f(x) - h(x)\| \leq \frac{\varphi_1(x, x, (1 - n)x)}{2(1 - L_1)} + \varphi_2(0, 0)
\]
for all $x \in X$.

By the similar argument to Theorem 2.1, one observes that the additive mapping $h$ yields $\mathbb{C}$-linear mapping. In addition, we figure out that
\[
\|h(ab) - ah(b) - h(a)b\| = \lim_{k \rightarrow \infty} \frac{1}{2^{2k}} \|D_2f(2^k a, 2^k b)\| \leq \lim_{k \rightarrow \infty} \frac{1}{2^{2k}} \varphi_1(0, 0, 0) + \varphi_2(2^k a, 2^k b) = 0
\]
for all $a, b \in X$. Thus, the mapping $h$ is linear derivation on $X$.

The remaining proof is similar to the corresponding part of Theorem 2.1. 

Based on the above theorem, we get the following corollary.

**Corollary 3.2.** Let $\xi : [0, \infty) \rightarrow [0, \infty)$ be a nontrivial function satisfying
\[
\xi(2t) \leq \xi(2)\xi(t), \quad (t \geq 0), \quad 0 < \xi(2) < 2.
\]
If $f : X \rightarrow X$ with $f(0) = 0$ is a mapping satisfying the functional inequality
\[
\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \theta_1 \xi(\|x\|) + \theta_2 \xi(\|y\|) + \theta_3 \xi(\|z\|) + \theta_4 \xi(\|a\|) \xi(\|b\|)
\]
for all $x, y, z, a, b \in X$ and for some $\theta_i \geq 0 (i = 1, \ldots, 4)$, then there exists a unique linear derivation $h : X \rightarrow X$ such that
\[
\|f(x) - f(0) - h(x)\| \leq \frac{(\theta_1 + \theta_2) \xi(\|x\|) + \theta_3 \xi(1 - n\|x\|)}{2 - \xi(2)} + \theta_4 \xi(0)^2,
\]
for all $x \in X$. 

Proof. Applying Theorem 3.1 with $L_1 := \frac{\xi(2)}{2} < 1$ and $L_2 := L_1^2$, we obtain the desired result. \qed

Theorem 3.3. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality

$$\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \varphi_1(x, y, z) + \varphi_2(a, b)$$

and there exist constants $L_1, L_2$ with $0 < L_1, L_2 < 1$ for which the functions $\varphi_1 : X^3 \to \mathbb{R}^+$ and $\varphi_2 : X^2 \to \mathbb{R}^+$ satisfy the property

$$\varphi_1\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L_1}{2} \varphi(x, y, z), \quad (3.5)$$

$$\varphi_2\left(\frac{a}{2}, \frac{b}{2}\right) \leq \frac{L_2}{4} \varphi(a, b)$$

for all $x, y, z, a, b \in X$. Then there exists a unique linear derivation $h : X \to X$ defined by $h(x) = \lim_{k \to \infty} 2^k f\left(\frac{x}{2^k}\right)$ such that

$$\|f(x) - h(x)\| \leq \frac{L_1 \varphi_1(x, x, (1 - n)x)}{2(1 - L_1)}$$

for all $x \in X$.

Proof. We observe that $f(0) = 0$ because $\varphi_1(0, 0, 0) = 0 = \varphi_2(0, 0)$ by the assumption (3.5). It follows from (3.4) and (3.5) that

$$\left\|f(x) - 2^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{i=0}^{m-1} 2^i \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \frac{(1 - n)x}{2^{i+1}}\right)$$

$$\leq \sum_{i=0}^{m-1} \frac{L_1^{i+1}}{2} \varphi(x, x, (1 - n)x)$$

$$\leq \frac{L_1}{1 - L_1} \frac{\varphi(x, x, (1 - n)x)}{2}$$

for all nonnegative integer $m$ and all $x \in X$.

The remaining proof is similar to the corresponding part of Theorem 3.1. \qed

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. Let $\xi : [0, \infty) \to [0, \infty)$ be a nontrivial function satisfying

$$\xi\left(\frac{t}{2}\right) \leq \xi\left(\frac{1}{2}\right) \xi(t), \quad (t \geq 0), \quad 0 < \xi\left(\frac{1}{2}\right) < \frac{1}{2}.$$ 

If $f : X \to Y$ is a mapping satisfying the functional inequality

$$\|D_1 f(\lambda x, y, z) - D_2 f(a, b)\| \leq \theta_1 \xi(||x||) + \theta_2 \xi(||y||) + \theta_3 \xi(||z||) + \theta_4 \xi(||a||) \xi(||b||)$$

for all $x, y, z, a, b \in X$. Then there exists a unique linear derivation $h : X \to X$ defined by $h(x) = \lim_{k \to \infty} 2^k f\left(\frac{x}{2^k}\right)$ such that

$$\|f(x) - h(x)\| \leq \frac{L_1 \theta_1 \xi(||x||) + \theta_2 \xi(||y||) + \theta_3 \xi(||z||) + \theta_4 \xi(||a||) \xi(||b||)}{2(1 - L_1)}$$

for all $x \in X$.\qed
for all \(x, y, z, a, b \in X\) and for some \(\theta_i \geq 0 (i = 1, \ldots, 4)\), then there exists a unique linear derivation \(h : X \to X\) such that

\[
\|f(x) - h(x)\| \leq \frac{\xi\left(\frac{1}{2}\right)}{1 - 2\xi\left(\frac{1}{2}\right)}\left[(\theta_1 + \theta_2)\xi(\|x\|) + \theta_3\xi(1 - n\|x\|)\right]
\]

\[
= \frac{(\theta_1 + \theta_2)\xi(\|x\|) + \theta_3\xi(1 - n\|x\|)}{\xi\left(\frac{1}{2}\right) - 2}
\]

for all \(x \in X\).

**Proof.** Applying Theorem 3.3 with \(L_1 := 2\xi\left(\frac{1}{2}\right) < 1\) and \(L_2 := L_1^2\), we lead to the approximation. \(\square\)

**References**


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