Presić Type Extension in Cone Metric Space

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Abstract. Let \((X, d)\) be a complete cone metric space, \(k\) a positive integer and \(T\) a mapping of \(X^k\) into \(X\). In this paper we prove that if \(T\) satisfies conditions (1) and (2) below, then there exists a unique \(x\) in \(X\) such that \(T(x, x, ..., x) = x\). Also, we have investigated under what conditions the mappings \(T : X^k \rightarrow X\) and \(f : X \rightarrow X\) have a common fixed point. Our results extend and generalize the results of [3],[4],[5],[9] and [10].

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Huang and Zhang [5] have replaced the real numbers by ordered Banach spaces and defined a cone metric space. They have proved some fixed point theorems for contractive mappings defined on these spaces. Further results on fixed point theorems in such spaces were obtained by several other mathematicians, see [1], [2], [6], [7], [8], [11], [12] and [13].

The following concepts are borrowed from [5].

Let $E$ be a real Banach space, and $P$ a subset of $E$. Then $P$ is called a cone if

(i) $P$ is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$,

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in \text{int } P$ where $\text{int } P$ denotes the interior of $P$. The cone $P$ is called normal if there is a number $K > 0$ such that for $x, y \in E$

$0 \leq x \leq y$ implies $||x|| \leq K||y||$

**Definition 1.1.** Let $X$ be a non-empty set. Let $\leq$ be a partial ordering on $E$ with respect to a normal cone $P$. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $0 < d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and the pair $(X, d)$ is called a cone metric space.

Let $\rho(x, y) = ||d(x, y)||$. Then a sequence $\{x_n\}$ in $X$ is said to be convergent to $x \in X$ if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x) \ll c$, for every $n \geq n_0$, and is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $\rho(x_m, x_n) \ll c$, for every $m, n \geq n_0$. A cone metric space $(X, d)$ is called a complete cone metric space if every Cauchy sequence in $X$ is convergent to a point of $X$. A self-map $T$ on $X$ is said to be continuous if $\lim_{n \to \infty} x_n = x$ implies that $\lim_{n \to \infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$ in $X$.

It should be noted here that if $\text{int } P = \{0\}$, then the concept of convergence and all related notions have no meaning. In the case $\text{int } P \neq \{0\}$, the above notion of convergence coincides with the usual convergence in the metric $\rho(x, y) = ||d(x, y)||$.

The following is an example of a cone metric space, see [5].
Example 1.2. Let $E = \mathbb{R}^2, P = \{(x, y) \in Ex : y \geq 0\}, X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then $P$ is a normal cone and $(X, d)$ is a cone metric space.

The objective of this paper is to proving fixed and common point point theorems for a certain class of maps in a cone metric space.

2. Fixed Points

Now, we formulate our main theorem.

Theorem 2.1. Let $(X, d)$ be a complete cone metric space, and $P$ be a normal cone with normal constant $K$. Suppose that $k$ is a positive integer and $T : X^k \to X$ is a mapping satisfying the following contractive type condition:

$$
\|d(T(x_1, x_2, x_3, ..., x_k), T(x_2, x_3, ..., x_{k+1}))\|
\leq \lambda \max\{\|d(x_i, x_{i+1})\| : 1 \leq i \leq k\}
$$

where $\lambda \in (0, 1)$ is a constant and $x_1, x_2, ..., x_{k+1}$ are arbitrary elements in $X$. Then

(i) there exists a point $x$ in $X$ such that $T(x, x, ..., x) = x$.

(ii) If $x_1, x_2, x_3, ..., x_k$ are arbitrary points in $X$ and for $n \in \mathbb{N}$,

$$
x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}),
$$

then the sequence $\{x_n\}^\infty_1$ converges to some $x$ in $X$ and $T(x, x, ..., x) = x$.

(iii) If, in addition, we suppose that on the diagonal $\Delta \subset X^k$, the inequality

$$
d(T(u, ..., u), T(v, ..., v)) < d(u, v)
$$

holds for all $u, v \in X$, with $u \neq v$, then $x$ is the unique point in $X$ such that $T(x, x, ..., x) = x$.

Proof. Let $x_1, x_2, ..., x_k$ be $k$ arbitrary points in $X$. Using these points define a sequence $\{x_n\}$ as follows:

$$
x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...).
$$

For simplicity set $\alpha_n = d(x_n, x_{n+1})$. We shall prove by induction that for each $n \in \mathbb{N}$:

$$
\|\alpha_n\| \leq R\theta^n, \text{where } \theta = \lambda^{1/k} \text{ and } R = \max\{\|\alpha_i\|/\theta^i : 1 \leq i \leq k\}.
$$

According to the definition of $R$ we see that (3) is true for $n = 1, ..., k$. Now let the following $k$ inequalities:

$$
\|\alpha_{n+i}\| \leq R\theta^{n+i}, 0 \leq i \leq k - 1
$$
be the induction hypothesis. Then we have:

\[
\|\alpha_{n+k}\| = \|d(x_{n+k}, x_{n+k+1})
\leq \|d(T(x_{n+1}, x_{n+2}, \ldots, x_{n+k+1}), T(x_{n+2}, x_{n+3}, \ldots, x_{n+k+2}))\|
\leq \lambda \max \{\|\alpha_n\|, \|\alpha_{n+1}\|, \ldots, \|\alpha_{n+k-1}\|\}
\leq \lambda \max \{R\theta^n, R\theta^{n+1}, \ldots, R\theta^{n+k-1}\}
= \lambda R\theta^n \{\text{as } \theta < 1\}
= R\theta^{n+k} \{\text{as } \lambda = \theta^k\}
\]

and the inductive proof of (3) is complete. Next using (3) for any \(n, p \in \mathbb{N}\) we have the following argument:

\[
d(x, x_{n+p}) \leq d(x, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})
\leq R\theta^n + R\theta^{n+1} + \ldots + R\theta^{n+p-1}
\leq R\theta^n (1 + \theta + \theta^2 + \ldots)
= R\theta^n / (1 - \theta)
\]

This means that

\[
\|d(x, x_{n+p})\| \leq K \|R\theta^n / (1 - \theta)\|
\]

which implies that \(\lim_{n,p \to \infty} \|d(x, x_{n+p})\| = 0\). So \(\lim_{n,p \to \infty} d(x, x_{n+p}) = 0\). Therefore, we conclude that \(\{x_n\}\) is a Cauchy sequence. Since \(X\) is a complete space, there exists \(x\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\). Then for any integer \(n\) we have:

\[
d(x, T(x, \ldots, x)) \leq d(x, x_{n+k}) + d(x_{n+k}, T(x, \ldots, x))
\leq d(x, x_{n+k}) + d(T(x, \ldots, x_{n+k-1}), T(x, \ldots, x))
\leq d(x, x_{n+k}) + d(T(x, \ldots, x, x), T(x, \ldots, x, x)) +
\quad T(x, \ldots, x, x_{n+k-1}) + \ldots
\quad d(T(x, x_{n+k-1}, x_{n+k-2}), T(x, x_{n+k-1}, x_{n+k-2}))
\leq d(x, x_{n+k}) + \lambda d(x, x_n) + \lambda \max \{d(x, x_n), d(x, x_{n+1}), \ldots, d(x_{n+k-1}, x_{n+k-1})\}.
\]

From this we obtain

\[
\|d(x, T(x, \ldots, x))\| \leq K(\|d(x, x_{n+k})\| + \lambda \|d(x, x_n)\| +
\lambda \max \{\|d(x, x_n)\|, \|d(x, x_{n+1})\|\} + \ldots
\lambda \max \{\|d(x, x_n)\|, \|d(x, x_{n+1})\|, \ldots, \|d(x_{n+k-2}, x_{n+k-1})\|\}) \to 0.
\]

Hence \(\|d(x, T(x, \ldots, x))\| = 0\). This implies that \(T(x, \ldots, x) = x\). Thus we proved (i) and (ii).

For the uniqueness of a fixed point, let \(y\) be another fixed point of \(T\). From (2), \(d(x, y) = d(T(x, x, \ldots, x), T(y, y, \ldots, y)) < d(x, y)\), which implies that \(d(x, y) = 0\). Hence \(x = y\). This completes the proof.
Remark 2.2. Theorem 2.1 is a generalization of the Theorem 1 of [5], if $k = 1$ in condition (1).

Corollary 2.3. If condition (1) in Theorem 2.1 is replaced by the condition

$$d(T(x_1, x_2, x_3, ..., x_k), T(x_2, x_3, ..., x_{k+1}))$$

(4)

$$\leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1}),$$

for every $x_1, ..., x_{k+1}$ in $X$, where $q_1, q_2, ..., q_k$ are non-negative constants such that $q_1 + q_2 + ... + q_k < 1$, then all the conclusions of Theorem 2.1 hold.

Proof. Since

$$q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1})$$

$$\leq (q_1 + q_2 + ... + q_k) \max\{d(x_1, x_2), d(x_2, x_3), ..., d(x_k, x_{k+1})\},$$

and $q_1 + q_2 + ... + q_k < 1$, then proof follows.

3. Common Fixed Points

Let $(X, d)$ be a cone metric space, $k$ a positive integer, $T : X^k \to X$ and $f : X \to X$ be mappings. We will investigate the conditions under which the mappings $f$ and $T$ will have a common fixed point.

An element $x \in X$ is said to be a coincidence point of $f$ and $T$ if $f(x) = T(x, x, ...x)$. $x$ is a common fixed point of $f$ and $T$ if $x = f(x) = T(x, x, ...x)$. The set of coincidence points of $f$ and $T$ is denoted by $C(f, T)$.

Definition 3.1. Mappings $f$ and $T$ are said to be commuting if $f(T(x, x, ...x)) = T(f(x), f(x), ...f(x))$ for all $x \in X$ and said to be coincidentally commuting if they commute at their coincidence points. Further, $f$ will be called idempotent at some $u \in C(f, T)$ if $f(f(u)) = f(u)$.

Theorem 3.2. Let $(X, d)$ be a normal cone metric space. Suppose $k$ is a positive integer, $T : X^k \to X$ and $f : X \to X$ be mappings satisfying the following conditions:

$$T(X^k) \subset f(X)$$

(5)

$$\|d(T(x_1, x_2, x_3, ..., x_k), T(x_2, x_3, ..., x_{k+1}))\|$$

(6)

$$\leq \lambda \max\{\|d(f(x_i), f(x_{i+1}))\| : 1 \leq i \leq k\},$$

where $\lambda \in (0, 1)$ is a constant and $x_1, x_2, ..., x_{k+1}$ are arbitrary elements in $X$, and $(f(X), d)$ is complete. Then $C(f, T) \neq \emptyset$. Further, if $f$ is idempotent at some $u \in C(f, T)$ and $f$ and $T$ are coincidentally commuting then $f$ and $T$ have a unique common fixed point.
Proof. Let \( x_1, x_2, \ldots, x_k \) be arbitrary elements in \( X \). We define sequence \( \{x_n\} \) in \( f(X) \) as follows:
\[
y_{n+k} = f(x_{n+k}) = T(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad \text{for} \quad n = 1, 2, \ldots
\]

Let \( \alpha_n = d(y_n, y_{n+1}) \). By the method of induction, we shall prove that

\[
(7) \quad \|\alpha_n\| \leq R\theta^n, \quad \text{where} \quad \theta = \lambda^{1/k} \quad \text{and} \quad R = \max \frac{\|\alpha_i\|}{\theta^i} : 1 \leq i \leq k.
\]

Clearly, from the definition of \( R \), (7) is true for \( n = 1, 2, \ldots, k \). Let the \( k \) inequalities \( \|\alpha_n\| \leq R\theta^n, \|\alpha_{n+1}\| \leq R\theta^{n+1}, \ldots, \|\alpha_{n+k-1}\| \leq R\theta^{n+k-1} \) be the induction hypothesis. Then we have

\[
\|\alpha_{n+k}\| = \|d(y_{n+k}, y_{n+k+1})\| = \|d(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots, x_{n+k}))\| \leq \lambda \max\{\|d(f(x_n), f(x_{n+1}))\|, \|d(f(x_{n+1}), f(x_{n+2}))\|, \ldots, \|d(f(x_{n+k-1}), f(x_{n+k}))\|\} = \lambda \max\{\|\alpha_n\|, \|\alpha_{n+1}\|, \ldots, \|\alpha_{n+k-1}\|\} \leq \lambda \max\{R\theta^n, R\theta^{n+1}, \ldots, R\theta^{n+k-1}\} = \lambda R\theta^n \quad \{\text{as} \quad \theta < 1\} \\
= R\theta^{n+k} \quad \{\text{as} \quad \lambda = \theta^k\}
\]

Thus inductive proof of (7) is complete. Now for \( n, p \in \mathbb{N} \), we have

\[
\|d(y_n, y_{n+p})\| \leq \|d(y_n, y_{n+1})\| + \|d(y_{n+1}, y_{n+2})\| + \ldots + \|d(y_{n+p-1}, y_{n+p})\| \\
\leq R\theta^n + R\theta^{n+1} + \ldots + R\theta^{n+p-1} \\
\leq R\theta^n(1 + \theta + \theta^2 + \ldots) \\
= R\theta^n/(1 - \theta)
\]

It means that

\[
\|d(y_n, y_{n+p})\| \leq K\|R\theta^n/(1 - \theta)\|.
\]

Then \( \lim_{n \to \infty} \|d(y_n, y_{n+p})\| = 0 \). Therefore, the sequence \( \{y_n\} \) is a Cauchy sequence in \( f(X) \). As \( (f(X), d) \) is complete there exists \( x \in f(X) \) such that \( \lim_{n \to \infty} y_n = x \). Let \( x = f(u) \) for some \( u \in X \). Then we have

\[
d(f(u), T(u, u, \ldots)) \leq d(f(u), y_{n+k}) + d(y_{n+k}, T(u, u, \ldots, u)) \\
= d(f(u), y_{n+k}) + d(T(x_n, x_{n+1}, \ldots, x_{n+k-1}), T(u, u, \ldots, u)) \\
\leq d(f(u), y_{n+k}) + d(T(u, u, \ldots, u), T(u, u, \ldots, x_n)) \\
+ \ldots + d(T(u, x_n, \ldots, x_{n+k-2}), T(x_n, x_{n+1}, \ldots, x_{n+k-1})) \\
\leq d(f(u), y_{n+k}) + \lambda d(f(u), f(x_n)) + \lambda \max\{d(f(u), f(x_n)), d(f(x_n), f(x_{n+1}))\} \\
+ \ldots + \lambda \max\{d(f(u), f(x_n)), d(f(x_n), f(x_{n+1})), \ldots, d(f(x_{n+k-2}), f(x_{n+k-1}))\}
\]
This yields

\[ ||d(f(u), T(u, u, \ldots u))|| \leq K(||d(f(u), y_{n+k})|| + \lambda ||d(f(u), f(x_n))||
+ \lambda \max\{||d(f(u), f(x_n))||, ||d(f(x_n), f(x_{n+1}))||\} + \ldots
+ \lambda \max\{||d(f(u), f(x_n))||, ||d(f(x_n), f(x_{n+1}))||, \\
..., ||d(f(x_{n+k-2}), f(x_{n+k-1}))||}\to 0.\]

Then \( ||d(f(u), T(u, u, \ldots u))|| = 0 \), which implies that \( f(u) = T(u, u, \ldots u) \),
\( i.e. \ C(f, T) \neq \emptyset \). Now, suppose \( f \) is coincidentally idempotent at \( u \),
and \( f \) is coincidentally commuting with \( T \). Then we have \( f(f(u)) = f(u) \) and \( f(T(u, u, \ldots u)) = T(f(u), f(u), \ldots f(u)) \). Therefore, we have \( f(u) = f(f(u)) = f(T(u, u, \ldots u)) = T(f(u), f(u), \ldots f(u)) \). Thus, we see that \( f(u) \) is a common fixed point of \( f \)
and \( T \), and \( \lim_{n \to \infty} y_n = f(\lim_{n \to \infty} y_n) = T(\lim_{n \to \infty} y_n, \lim_{n \to \infty} y_n, \ldots \lim_{n \to \infty} y_n) \).

Now, suppose \( x \) and \( y \) are two fixed points of \( f \) and \( T \) with \( x \neq y \). Then we have

\[ d(x, y) = d(T(x, x, \ldots x), T(y, y, \ldots y)) < d(f(x), f(y)) = d(x, y), \] a contradiction. Hence \( x = y \).

**Remark 3.3.** If \( f \) is the identity mapping on \( X \), we get Theorem 2.1.

**Remark 3.4.** Theorem 3.2 is an extension and generalisation of results of [3],[4],[9] and [10].

**Example 3.5.** Let \( E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\}, X = [0, 1], \) and \( d : X \times X \to E \) such that \( d(x, y) = (|x-y|, |x-y|) \). Then \( d \) is a cone metric on \( X \), (see Example 1.2). Define \( T : X^2 \to X \) and \( f : X \to X \) as follows:

\[ T(x, y) = \frac{x^2 + y^2}{4} + \frac{1}{2} \quad \text{and} \quad f(x) = x^2. \]

Then

\[ d(T(x, y), T(y, z)) = (\frac{|x^2 - y^2|}{4}, |x^2 - y^2|) \leq \frac{1}{2} \max\{d(f(x), f(y)), d(f(y), f(z))\}. \]

Further, 1 is a coincident point of \( f \) and \( T \). Also, \( f(f(1)) = f(1) \) and \( f \) and \( T \) commute at 1. Finally, 1 is the unique common fixed point of \( f \) and \( T \).

**Example 3.6.** Let \( X = \mathbb{R}, E = C_\mathbb{R}^1[0, 1] \) and \( P = \{\varphi \in E : \varphi \geq 0\} \). Define \( d : X \times X \to E \) by \( d(x, y) = |x - y| \varphi \), where \( \varphi : [0, 1] \to \mathbb{R} \) is such that \( \varphi(t) = e^t \). It is easy to see that \( d \) is a cone metric on \( X \). Consider the mappings \( T : X^2 \to X \) and \( f : X \to X \) in the following manner:

\[ T(x, y) = \frac{x + y}{1 + \alpha} \quad \text{and} \quad f(x) = 2x, \]

where \( \alpha > 1 \). One can see that

\[ d(T(x, y), T(y, z)) \leq k \max\{d(fx, fy), dfy, f(z)\}. \]
for all $x, y \in X$, where $k = \frac{1}{\alpha} \in (0, 1)$. The mappings $f$ and $T$ commute at $x = 0$. Finally, $x = 0$ is the unique common fixed point of $f$ and $T$.

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**References**


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