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On Generalized gp*- Closed Set in Topological Spaces

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Abstract

In this paper, the authors introduce a new class of sets called gp*-closed sets in topological spaces. Also we discuss some of their properties and investigate the relations between the associated topology.

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1. Introduction

In 1970, Levine Introduced the concept of generalized closed set and discussed the properties of sets, closed and open maps, compactness, normal and separation axioms. A.S.Mashor Abd.El-Monsef.M.E and Ei-Deeb.S.N., introduced pre-open set and investigated the properties of topology. Later in 1998 H.Maki, T.Noiri gave a new type of generalized closed sets in topological space called gp- closed sets. The investigation on generalization of closed set has lead to significant contribution to the theory of separation axiom, generalization of continuity and irresolute functions. The authors introduce new type of closed set named 'gp*-closed set' in the topological spaces.

. The aim of this paper is to continue the study of gp*-closed sets thereby contributing new innovations and concepts in the field of topology through analytical as well as research works. The notion of gp*-closed sets and its different characterizations are given in this paper. Throughout this paper (X,τ) and (Y,σ) represents the non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $A\subseteq X$, the closure of A and interior of A will be denoted by cl(A) and int(A) respectively.

2. Preliminaries

Definition 2.1[4]: Let A subset A of a topological space (X,τ), is called a pre-open set if $A \subseteq$ Int(cl (A)).

Definition 2.2 [1]: Let A subset A of a topological space (X,τ), is called a generalized closed set (briefly g-closed) if cl (A) $\subseteq U$ whenever $A\subseteq U$ and U is open in X.

Definition 2.3 [5]: Let A subset A of a topological space (X,τ), is called a generalized pre- closed set (briefly gp- closed) if pcl (A) $\subseteq U$ whenever $A\subseteq U$ and U is open in X.

Definition 2.4 [3]: Let A subset A of a topological space (X,τ), is called a generalized pre- closed set (briefly pg- closed) if pcl (A) $\subseteq U$ whenever $A\subseteq U$ and U is pre-open in X.

Definition 2.5 [8]: Let A subset A of a topological space (X, τ), is called a generalized pre- closed set (briefly g*- closed) if cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in X.

Definition 2.6 [10]: Let A subset A of a topological space (X, τ), is called a generalized pre- closed set (briefly g*p- closed) if pcl (A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in X.

Definition 2.7 [7]: Let A subset A of a topological space (X, τ), is called a generalized pre- closed set (briefly strongly g- closed) if cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is g-open in X.

Definition 2.8 [6]: Let A subset A of a topological space (X, τ), is called a generalized pre- closed set (briefly α g- closed) if α cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

Definition 2.9 [9]: Let A subset A of a topological space (X, τ), is called a generalized pre- closed set (briefly g#- closed) if cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is α g-open in X.

3. gp*- closed sets and gp*-open sets

Definition 3.1 A subset A of a topological space (X, τ), is called gp*-closed set if cl (A) $\subseteq$ U whenever A $\subseteq$ U and U is gp open in X.

Theorem 3.2. Every closed set is gp*-closed.
Proof. Let A be any closed set and U be any gp-open set containing cl(A) $\subseteq$ A = U. Hence A is gp*-closed set in X.

The converse of above theorem need not be true as seen from the following example.
Example 3.3. Let X = {a,b,c} with τ = {X, φ,{a},{c},{a,c},{b,c}}.The set {a,b} is gp*-closed set but not a closed set.

Theorem 3.4. Every g*-closed set is gp*-closed set.
Proof. Let A be any g*-closed set in X and U be any g-open set containing A. Since any g-open set is gp-open, Therefore cl(A) $\subseteq$ U. Hence A is gp*-closed set.

The converse of above theorem need not be true as seen from the following example.
Example 3.5. Let X = {a,b,c} with τ = {X, φ, {a},{a,b},{a,c}}.The set {a,c} is gp*-closed set but not a g*- closed set.

Theorem 3.6. Every pg -closed set is gp*-closed set.

Proof. Let A be any pg-closed set in X and U be pre-open set containing A. Since every pre-open set is gp-open set, we have pcl(A)$\subseteq$ cl(A) $\subseteq$ U. Therefore cl(A) $\subseteq$ U. Hence A is gp*-closed set.
The converse of above theorem need not be true as seen from the following example.

Example 3.7 Let X = {a,b,c} with τ = {X, φ, {a},{a,b},{a,c}}.The set {a} is gp*-closed set but not a pg- closed set.

Theorem 3.8. Every g*p-closed set is gp*-closed set.

Proof. Let A be any g*p-closed set in X such that U be g-open set containing A. Since every g-open set is gp-open set, we have pcl(A)$\subseteq$cl(A) $\subseteq$ U. Therefore cl(A) $\subseteq$ U. Hence A is gp*-closed set.
The converse of above theorem need not be true as seen from the following example.

Example 3.9 Let X = {a,b,c} with τ = {X,φ,{,c}}. The set {b} is gp*-closed set but not a g*p-closed set.

Theorem 3.10. Every gp*-closed set is strongly g-closed set.

Proof. Let A be any gp*-closed set in X such that U be any g- open set containing A. Since every g-open set is gp-open set, we have cl(A)$\subseteq$U. Hence A is strongly g-closed set.
The converse of above theorem need not be true as seen from the following example.

Example 3.11. Let X = {a,b,c} with τ = {X,φ, {c},{b,c}}. The set {a,c} is strongly g-closed set but not a gp*-closed set.

Theorem 3.12. Every gp*-closed set is g#-closed set.
Proof. Let A be any gp*-closed set in X such that U be any α g-open set containing A. Since every α g-open set is gp-open set. Therefore cl(A)$\subseteq$U. Hence A is g#-closed set.
The converse of above theorem need not be true as seen from the following example.

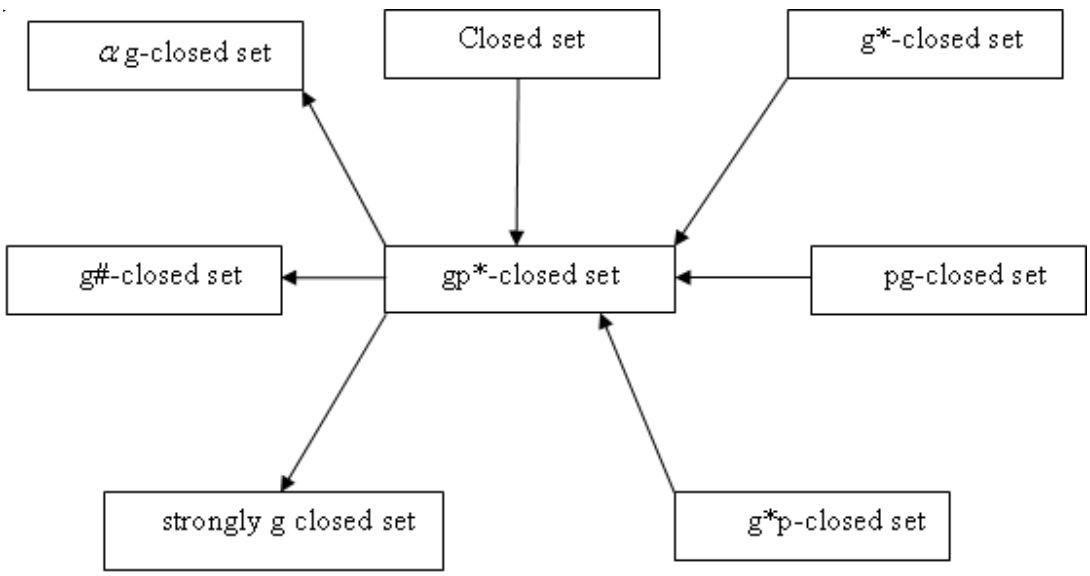
Example 3.13. Let X = {a,b,c} with τ = {X, φ,{b}}. The set {b,c} is g#- closed set but not a gp*-closed set.

Theorem 3.14. Every gp*-closed set is α g*-closed set.

Proof. Let A be any gp*-closed set in X such that U be any α g-open set containing A. Since every α g-open set is gp-open set. Therefore cl(A)$\subseteq$U. Hence A is α $\mathbf{g}^*$ -closed set.
The converse of above theorem need not be true as seen from the following example.

Example 3.15. Let X = {a,b,c} with τ = {X, φ,{a,b}}. The set {a,c} is α $\mathbf{g}^*$ - closed set but not a gp*-closed set.

Remark: 3.16: By the above results we have the following diagram:



A B means A imply B.

Definition 3.2. A subset A of a topological space (X,τ), is called gp*- open set if and only if A^c is gp*-closed in X. We denote the family of all gp*-open sets in X by gp*-O(X).

Theorem 3.15 (i) Every open set is gp*-open.
(ii) Every g^*-open set is gp*-open set.
(iii) Every pg -open set is gp*-open set.
(iv) Every g*p-open set is gp*-open set.
(v) Every gp*-open set is strongly g-open set.
(vi) Every gp*-open set is g#-open set.
(vii) Every gp*-open set is α g^*-open set.

4. Characteristics of gp*-closed and gp*-open sets

Theorem 4.1. If A and B are gp*-closed sets in X then $A \cup B$ is gp*-closed set in X.

Proof. Let A and B are gp*-closed sets in X and U be any gp-open set containing A and B. Therefore cl(A) $\subseteq$U, cl (B)$\subseteq$U. Since A $\subseteq$U, B $\subseteq$U then $A \cup B$ $\subseteq$U. Hence cl ($A \cup B$) = cl (A)$\cup$ cl (B) $\subseteq$U. Therefore $A \cup B$ is gp*-closed set in X.

Theorem 4.2. If a set A is gp*-closed set iff cl (A)-A contains no non empty gp-closed set.
Proof.

Necessity: Let F be a gp-closed set in X such that F $\subseteq$ cl (A) –A. Then A $\subseteq$ X – F. Since A is gp* closed set and X - F is gp-open then cl (A) $\subseteq$ X – F. (i.e.) F$\subseteq$X –cl (A). So F$\subseteq$(X –cl (A)) $\cap$ (cl (A) –A). Therefore F = φ

Sufficiency: Let us assume that cl (A) – A contains no non empty semi-closed set. Let $A \subseteq U$, U is semi open. Suppose that cl(A) is not contained in U, cl(A) $\cap$ U^c is a non-empty gp- closed set of cl(A)-A which is contradiction. Therefore cl(A)$\subseteq$U. Hence A is gp*-closed.

Theorem 4. 3. The Intersection of any two subsets of gp*-closed sets in X is gp*-closed set in X.
Proof. Let A and B are any two sub sets of gp*-closed sets. A $\subseteq$U, U is any gp-open and B$\subseteq$U, U is gp-open. Then cl (A) $\subseteq$U, cl (B) $\subseteq$U, therefore cl ($A \cap B$) $\subseteq$U, U is gp-open in X. Since A and B are gp*-closed set, Hence $A \cap B$ is a gp*-closed set.

Theorem 4.4 If A is gp*-closed set in X and A $\subseteq$ B $\subseteq$ cl (A), Then B is gp*-closed set in X.

Proof. Since B$\subseteq$cl(A), we have cl(B) $\subseteq$ cl(A) then cl(B) - B $\subseteq$ cl(A) – A. By theorem 4.2, pcl (A)-A contains no non empty gp-closed set. Hence cl (B) - B contains no non empty gp- closed set. Therefore B is gp*-closed set in X.

Theorem 4.5. If A $\subseteq$ Y $\subseteq$ X and suppose that A is gp* closed set in X then A is gp*-closed set relative to Y.

Proof. Given that $A \subseteq Y \subseteq X$ and A is gp*-closed set in X. To prove that A is gp*-closed set relative to Y. Let us assume that $A \subseteq Y \cap U$, where U is gp- open in X. Since A is gp*- closed set, $A \subseteq U$ implies cl (A)$\subseteq$ U. It follows that $Y \cap$ cl (A)$\subseteq Y \cap$ U. That is A is gp*- closed set relative to Y.

Theorem 4.6. If A is both gp-open and gp*-closed set in X, then A is gp-closed set.

Proof. Since A is gp-open and gp* closed in X, cl(A) $\subseteq$ U. But Always A $\subseteq$cl (A). Therefore A = cl (A). Hence A is gp-closed set.

Theorem 4.7. For $x \in X$, then the set X-{x} is a gp*-closed set or gp- open.

Proof. Suppose that X – {x} is not gp-open, then X is the only gp- open set containing X – {x}. (i.e.) cl(X-{x})$\subseteq$ X. Then X-{x} is gp*-closed in X.

Theorem 4.8: If A and B are gp*-open sets in a space X. Then $A \cap B$ is also gp*-open set in X.

Proof. If A and B are gp*-open sets in a space X. Then A^c and B^c are gp*-closed sets in a space X. By theorem 4.1 $A^c \cup B^c$ is also gp*-closed set in X. (i.e.) $A^c \cup B^c = (A \cap B)^c$ is a gp*-closed set in X. Therefore $A \cap B$ gp*-open set in X.

Remark 4.9. The union of two gp*-open sets in X is generally not a gp*-open set in X.

Example 4.10.: Let X = {a, b, c} with τ = {X,φ,{a},{c},{a,c},{b,c}}. If A = {b}, B = {a} are gp*-open sets in X, then $A \cup B$ = {a,b} is not gp* open set in X.

Theorem 4.11: If Int (B) $\subseteq$ B$\subseteq$A and if A is gp*-open in X, then B is gp*-open in X.

Proof. Suppose that Int (B) $\subseteq$B $\subseteq$A and A is gp*-open in X then $A^c \subseteq B^c \subseteq$Cl ($A^c$). Since A^c is gp*-closed in X, we have B is gp*-open in X.

Theorem 4.12. A set A is gp* open if and only if F$\subseteq$Int(A), where F is gp-closed and F $\subseteq$A.

Proof. If F$\subseteq$Int(A) where F is gp-closed and F$\subseteq$A. Let $A^c \subseteq G$ where G = F^c is gp-open. Then $G^c \subseteq A$ and $G^c \subseteq$Int(A). Then we have A^c is gp*-closed. Hence A is gp*-open.

Conversely If A is gp*-open, F$\subseteq$A and F is gp-closed. Then F^c is gp-open and $A^c \subseteq F^c$. Therefore cl(A^c)$\subseteq$(F^c). But cl(A^c) = $(\text{Int}(A))^c$. Hence F$\subseteq$Int(A).

Theorem 4.13. Let (X,τ) be a normal space and if Y is a gp*-closed subset of (X,τ), then the subspace Y is normal.
Proof. If G_1 and G_2 disjoint closed sets in (X,τ) such that $(Y\cap G_1)\cap(Y\cap G_2)=\varphi$. Then $Y\subseteq(G_1\cap G_2)^c$ and $\subseteq(G_1\cap G_2)^c$ is gp-open. Y is gp*-closed in (X,τ). Therefore $cl(Y)\subseteq(G_1\cap G_2)^c$. Hence $(cl(Y)\cap G_1)\cap(cl(Y)\cap G_2)=\varphi$. Since (X,τ) is normal, there exists disjoint open sets A and B such that $cl(Y)\cap G_1\subseteq A$ and $cl(Y)\cap G_2\subseteq B$. (i.e) $Y\cap A$ and $Y\cap B$ are open sets of Y such that $Y\cap G_1\subseteq Y\cap A$ and $Y\cap G_2\subseteq Y\cap B$ are disjoint open sets of Y such that $Y\cap G_1\subseteq Y\cap A$ and $Y\cap G_2\subseteq Y\cap B$. Hence Y is normal.

5. gp*-neighbourhoods

In this section we Introduce gp*-neighborhoods in topological spaces by using the notions of gp*-open sets and study some of their properties.

Definition 5.1: Let x be a poInt in a topological space X and let $x\in X$. A subset N of X is said to be a gp*-nbhd of x iff there exists a gp*-open set G such that $x\in G\subset N$.

Definition 5.2: A subset N of Space X is called a gp*-nbhd of $A\subset X$ iff there exists a gp*-open set G such that $A\subset G\subset N$.

Theorem 5.3: Every nbhd N of $x\in X$ is a gp*-nbhd of X.

Proof: Let N be a nbhd of point $x\in X$. To prove that N is a gp*-nbhd of x. By definition of nbhd, there exists an open set G such that $x\in G\subset N$. Hence N is a gp*-nbhd of x.

Remark 5.4. In general, a gp*-nbhd of $x\in X$ need not be a nbhd of x in X as seen from the following example.

Example 5.5. Let X = {a, b, c} with topology τ = {X, φ, {a},{c},{a,c},{b,c}}. Then gp*-O(X) = {X, φ,{a},{b},{c},{a,c},{b,c}}. The set {a,b} is gp*-nbhd of point a, since the gp*-open sets {a} is such that $a\in\{a\}\subset\{a,b\}$. However, the set {a,b} is not a nbhd of the poInt a, since no open set G exists such that $c\in G\subset\{a,b\}$.

Remark 5.6 The gp*-nbhd N of $x\in X$ need not be a gp*-open in X.

Example 5.7: Let X = {a, b, c} with topology τ = {X, φ,{a},{c}{a,c},{b,c}}. Then gp*-O(X) = {X, φ, {a},{b},{c},{b,c},{a,c}}. The set {a,b} is gp*-open sets {c}, but it is a gp*-nbhd if {c}. Since {c} is gp*-open set such that $c \in \{c\} \subset \{b,c\}$.

Theorem 5.8. If a subset N of a space X is gp*-open, then N is gp*-nbhd of each of its points.

Proof. Suppose N is gp*-open. Let $x \in N$. We claim that N is gp*-nbhd of x. For N is a gp*-open set such that $x \in N \subset N$. Since x is an arbitrary point of N, it follows that N is a gp*-nbhd of each of its points.

Remark 5.9: In general, a gp*-nbhd of $x \in X$ need not be a nbhd of x in X as seen from the following example.

Example 5.10: Let X = {a, b, c} with topology τ = {X, φ,{a},{c},{a,c},{b,c}}. Then gp*-O(X) = X, φ, {a}, {b},{c},{b,c},{a,c}}. The set {a,b} is gp*-nbhd of point b, since the gp*-open sets {b} is such that $b \in \{b\} \subset \{a,b\}$. Also the set {a,b} is gp*-nbhd of point {a}. Since the gp*-open set {a} is such that $a \in \{a\} \subset \{a,b\}$. (i.e.) {a,b} is gp*-nbhd of each of its points. However, the set {a,b} is not a gp*-open set in X.

Theorem 5.11. Let X be a topological space. If F is gp*-closed subset of X and $x \in F^c$. Prove that there exists a gp*-nbhd N of x such that $N \cap F = \varphi$

Proof: Let F be gp*-closed subset of X and $x \in F^{c.}$ Then F^c is gp*-open set of X. So by theorem 5.2 F^c contains a gp*-nbhd of each of its points. Hence there exists a gp*-nbhd N of x such that
$N \subset F^{c.}$. (i.e.) $N \cap F = \varphi$

Definition 5.12. Let x be a point in a topological space X. The set of all gp*-nbhd of x is called the gp*-nbhd system at x, and is denoted by gp*-N(x).

Theorem 5.13. Let a gp*-nbhd N of X be a topological space and each $x \in X$, Let gp*-N(X, τ) be the collection of all gp*-nbhd of x. Then we have the following results.

(i) $\forall x \in X$, gp*-N(x) $\neq \phi$.
(ii) $N \in$ gp*-N(x) $\Rightarrow x \in N$.
(ii) $N \in$ gp*-N(x), $M \supset N \Rightarrow M \in$ gp*-N(x).
(iii) $N \in$ gp*-N(x), $M \in$ gp*-N(x) $\Rightarrow N \cap M \in$ gp*-N(x).

(iv) $N \in$ gp*-N(x) $\Rightarrow$ there exists $M \in$ gp*-N(x) such that $M \subset N$ and $M \in$ gp*-N(y) for every $y \in M$.

Proof: (i) Since X is gp*-open set, it is a gp*-nbhd of every $x \in X$. Hence there exists at least one gp*-nbhd (namely-X) for each $x \in X$. Therefore gp*- N(x) $\neq \phi$ for every $x \in X$

(ii) If $N \in$ gp*-N(x), then N is gp*-nbhd of x. By definition of gp*-nbhd, $x \in N$.

(iii) Let $N \in$ gp*-N(x) and $M \supset N$. Then there is a gp*-open set G such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so M is gp*-nbhd of x. Hence $M \in$ gp*-N(x).

(iv) Let $N \in$ gp*-N(x), $M \in$ gp*-N(x). Then by definition of gp*-nbhd, there exists gp*- open sets G_1 and G_2 such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$.
Hence $x \in G_1 \cap G_2 \subset N \cap M$ -------- (1).
Since $G1 \cap G_2$ is a gp*-open set,(being the Intersection of two reg-open sets), it follows from (1) that $N \cap M$ is a gp*-nbhd of x. Hence $N \cap M \in$ gp*-N(x).

(v) Let $N \in$ gp*-N(x), Then there is a gp*-open set M such that $x \in M \subset N$. Since M is gp*-open set, it is gp*-nbhd of each of its points. Therefore $M \in$ gp*-N(y) for every $y \in M$.

CONCLUSION

The pg*-closed set can be used to derive continuity, closed map, open map and homeomorphism, closure and interior and new separation axioms.

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