Stability and Density of Periodic Points

of $L_p$-Space, $0 < p < 1$

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Abstract. In this paper we define a type of $L_p$-spaces for $p < 1$ and prove that it is Banach space in term of $L_p$-metric for $p < 1$. Then define some operators on it and study the chaos and stability of the sequence of these operators.

Keywords: von Forrester-Lasota equation, stability

1. Introduction

The problem of the chaotic behavior of partial differential equation was considered by [1],[2], Loskot, [5], and dense set of periodic solution of the problem [3], the semigroup $\{T_t\}_{t \geq 0}$ generated by differential equation has invariant measure with arbitrary large dimension 4. By [1]the problem chaotic and stability of equation $\tilde{L}_p$ space in the present paper we look the chaotic in $\tilde{L}_p$ for $0 < p < 1$.

2. Preliminaries

Definition 2.1: Complete quasi-normed space is called quasi Banach space.

Definition 2.2: We call semi-dynamical system any measurable application: $\Phi: [0, \infty] \times X \rightarrow X$
Which has the properties?
• \( \Phi(0, x) = x, \forall x \in X \)
• \( \Phi(s + t, x) = \Phi(x, \Phi(t, x)), \forall s, t \in [0, +\infty], \forall x \in X \)
• \( \Phi(t, x) = \omega, \text{and } t' > t \Rightarrow \Phi(t', x) = \omega \)
• \( \Phi(t, x) = \Phi(t, y), \forall t > 0 \Rightarrow x = y(X, \Box) \) denotes a measurable space, \( \omega \in X \) is a fixed point.

**Definition 2.3:** The indicator function of a subset \( A \) of a set \( X \) is a function

\[ 1_A : X \rightarrow \{0, 1\} \] defined as

\[ 1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \]

**Definition 2.4:** A semi group \( (T_t)_{t \geq 0} \) is an algebraic structure consisting of a set to gether with an associative binary operator.

**Definition 2.5:** The semi group \( (T_t)_{t \geq 0} \) is strongly stable in \( V \) if and only if for every \( v \in V, \lim_{t \to \infty} T_t v = 0 \) in \( V \)

**Definition 2.6:** The semi group \( (T_t)_{t \geq 0} \) is exponentially stable on \( V \) if and only if there exists \( D \leq \infty \) and \( \omega > 0 \) such that \( \|T_t\| \leq De^{-\omega t} \), for \( t \geq 0 \)

We consider the partial differential equation

\[ \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \gamma u, \quad t \geq 0, \quad 0 \leq x \leq 1 \]

with initial condition

\[ u(0, u) = v(x), \quad 0 \leq x \leq 1 \]

where \( v \) belongs to some normed vector space \( V \) of function defined on \([0,1]\).

The function \( \tilde{T}_t \) is given by the formula

\[ (\tilde{T}_t v)(x) = \tilde{u}(t, x) = e^{\gamma t}v(xe^{-t}), x \in [0,1] \]

Where \( \tilde{u} \) is the unique solution of (1) and (2).

### 3. The main result

In this article we shall introduce our main results:

**Proposition 3.1:** \( L_p(I) \) is complete quasi-norm for \( p < 1 \).

**Proof:**

\[
L_p(I) = \left\{ f : I \rightarrow \mathbb{R} : \left( \int_I |f(x)|^p \, dx \right)^{1/p} < \infty \right\}
\]

\[
\|f\|_p = \left( \int_I |f(x)|^p \, dx \right)^{1/p}
\]
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Let \( \{ f_n \} \) Cauchy sequence in \( L_p \) then for a given \( \varepsilon > 0 \) we can find \( k \in \mathbb{N} \)

\[
\| f_n - f_m \|_p < \varepsilon \quad \forall \ n, m > k
\]

This implies

\[
\frac{1}{L} \int I \left| f_n(x) - f_m(x) \right|^p dx < \varepsilon
\]

\( f_n(x) \in R, f_m(x) \in R \) i.e. \( \{ f_n(x) \} \) is Cauchy sequence in \( R \). Since \( R \) is complete. So \( \exists f_0(x) \in R \) such that

\[
\left| f_n(x) - f_0(x) \right| < \frac{\varepsilon}{|I|} \quad \forall n > k
\]

\[
\left( \int I \left| f_n(x) - f_0(x) \right|^p dx \right)^{\frac{1}{p}} < \left( \int I \varepsilon^p dx \right)^{\frac{1}{p}} = \frac{|\varepsilon|}{|I|} = \varepsilon
\]

Since \( R \) is complete. So \( L_p, p < 1 \) is complete quasi-norm.

To prove the following result we need, the following lemma.

**Lemma 3.2:** (Fatuous)

\[
\lim_{n \to \infty} \inf_{R} \int f_n d\mu \geq \int \lim_{Rn \to \infty} \inf d\mu
\]

**Definition 3.3:** Denote by \( \tilde{L}^p \) the space of all functions

satisfying the following condition:

\[
v \in L^p(0,1) (0 < p < 1)
\]

\[
S_A(v) = \sup_{x \in A} \left( \frac{1}{x_0} \int_{x_0}^{x} |v(s)|^p ds \right)^{\frac{1}{p}} < \infty
\]

**Proposition 3.4:** The space \( \tilde{L}_p \) with norm \( S_{[0,1]} \) is Banach space.

**Proof:** Let \( \{ v_n \} \) be a Cauchy sequence in \( \tilde{L}_p \) space. so for a given \( \varepsilon > 0 \) we can find \( n_0 \) such that for all numbers \( n, m \geq n_0 \), we can find
For all \( x \in (0,1] \)

\[
\left( \frac{1}{x} \int_{0}^{x} \|v_n(s) - v_m(s)\|^p \, ds \right)^{\frac{1}{p}} < \varepsilon
\]

By Fatou’s lemma \( \{v_n\} \) is also a Cauchy sequence in \( L_p \), according to the quasi norm. Because \( L_p \) is complete quasi norm, so there exist \( v_0 \in L_p \) such that

\[
v_n \to v_0 \quad \text{in} \quad L_p
\]

for any \( x \in (0,1] \) we have from Fatou’s lemma

\[
\left( \frac{1}{x} \int_{0}^{x} \|v_n(s) - v_0(s)\|^p \, ds \right)^{\frac{1}{p}} \leq \lim_{m \to \infty} \inf_{m \to \infty} \left( \frac{1}{x} \int_{0}^{x} \|v_n(s) - v_m(s)\|^p \, ds \right)^{\frac{1}{p}} < \varepsilon
\]

Therefore \( v_n \to v_0 \) as \( n \to \infty \), similarity, using Fatou’s lemma we can prove that \( v_0 \in \tilde{L}_p \).

**Proposition 3.5:** If \( 0 < a < b < c \leq 1 \), then

\[
S_{[a,c]} \leq S_{[a,b]} + S_{[b,c]}
\]

for \( 0 < p < 1 \).

**Proof:** If \( x \in (a,b] \) since \( 0 < a < b < c \leq 1, a, b, c > 0 \) then

\[
\left( \frac{1}{x} \int_{0}^{x} \|v(s)\|^p \, ds \right)^{\frac{1}{p}} > 0 \quad \forall x \in (a,b] , \text{this imply}
\]

\[
S_{[a,b]}, S_{[b,c]}, S_{[a,c]} > 0
\]

\[
\left( \frac{1}{x} \int_{0}^{x} \|v(s)\|^p \, ds \right)^{\frac{1}{p}} < \sup_{x \in [a,b]} \left( \frac{1}{x} \int_{0}^{x} \|v(s)\|^p \, ds \right)^{\frac{1}{p}} < S_{[a,b]} + S_{[b,c]}
\]

(1)

Then if \( x \in (b,c] \)

\[
\left( \frac{1}{x} \int_{0}^{x} \|v(s)\|^p \, ds \right)^{\frac{1}{p}} < \sup_{x \in [b,c]} \left( \frac{1}{x} \int_{0}^{x} \|v(s)\|^p \, ds \right)^{\frac{1}{p}}
\]

\[
S_{[a,b]} + S_{[b,c]}
\]

(2)

(1) and (2) imply
\[ S_{[a,c]} \leq S_{[a,b]} + S_{[b,c]} . \]

**Proposition 3.6:** If \( 0 < p_2 < p_1 < 1 \), then \( \tilde{L}_{p_1} \subset \tilde{L}_{p_2} \).

**Proof:**

Let \( v \in \tilde{L}_{p_1} \) then

\[
\left( \frac{1}{x} \int_0^x |v(s)|^{p_1} \, ds \right) \frac{1}{p_1} < \infty
\]

Since \( p_2 < p_1 \)

\[
\left( \frac{1}{x} \int_0^x |v(s)|^{p_2} \, ds \right) \frac{1}{p_2} \left( \frac{1}{x} \int_0^x |v(s)|^{p_1} \, ds \right) \frac{1}{p_1} < \infty \]

Then

\[
\left( \frac{1}{x} \int_0^x |v(s)|^{p_2} \, ds \right) < \infty \] then \( v \in \tilde{L}_{p_2} \).

**Definition 3.7:** Denote by \( \hat{L}_p \) the space of all function \( v \in \tilde{L}_p (0 < p < 1) \)

satisfying the following condition \( \lim_{a \to 0} S_{[0,a]} (v) = 0 \)

\[
\hat{L}_p = \left\{ v \in \tilde{L}_p (0 < p < 1) : \lim_{a \to 0} \sup_{x \in [0,a]} \left( \frac{1}{x} \int_0^x |v(s)|^p \, ds \right)^{1/p} < \infty \right\}
\]

**Proposition 3.8:** The space \( \hat{L}_p \) with the norm \( S_{[0,1]} \) is Banach space.

**Proof:** The space \( \tilde{L}_p \) with norm \( S_{[0,1]} \) is Banach space since \( \hat{L}_p \subset \tilde{L}_p \) and \( \hat{L}_p \) is linear subspace, it is sufficient to show that \( \hat{L}_p \) is its closed subspace. Let \( \{v_n\} \) a sequence in \( \hat{L}_p \).

Since \( v_n \) in \( \tilde{L}_p, \hat{L}_p \) Banach space then \( v_n \) is convergence Cauchy sequence.
Such that \[
\sup_{x \in (0,1]} \left( \frac{1}{x} \int_0^x |v_n(s) - v_\circ(s)|^p \, ds \right) \frac{1}{p} < \varepsilon
\]

We need to show that \( v_\circ \in \hat{L}_p \). Let \( \varepsilon > 0 \), there exists \( n_\circ \) such that \( \forall n \geq n_\circ \)

\[
\sup_{x \in (0,1]} \left( \frac{1}{x} \int_0^x |v_n(s) - v_\circ(s)|^p \, ds \right) \frac{1}{p} < \frac{\varepsilon}{2}
\]

\[
\sup \{|v_\circ| - \sup |v_n| \leq \sup \{ |v_\circ - v_n| \}
\]

\[
\sup |v_\circ| \leq \sup \{|v_\circ - v_n| \} + \sup |v_n|
\]

For every \( x < x_\circ \), we have

\[
\sup_{x < x_\circ} \left( \frac{1}{x} \int_0^x |v_\circ(s)|^p \, ds \right) \frac{1}{p} \leq \sup_{x < x_\circ} \left( \frac{1}{x} \int_0^x |v_n(s)|^p \, ds \right) \frac{1}{p}
\]

\[
\left( \frac{1}{x} \int_0^x |v_\circ(s)|^p \, ds \right) \frac{1}{p} \leq \sup_{x < x_\circ} \left( \frac{1}{x} \int_0^x |v_\circ(s)|^p \, ds \right) \frac{1}{p}
\]

\[
\leq \sup_{x < x_\circ} \left( \frac{1}{x} \int_0^x |v_\circ(s) - v_n(s)|^p \, ds \right) \frac{1}{p} + \sup_{x < x_\circ} \left( \frac{1}{x} \int_0^x |v_n(s)|^p \, ds \right) \frac{1}{p} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

4. Density of periodic point and stability of dynamical system \((\tilde{T}_t)_{t \geq 0}\)

**Theorem 4.1:** If \( \gamma > 0 \), then for any \( t_\circ \) there exist \( v_\circ \in \hat{L}_p \) for \( p < 1 \) such that
\[ \widehat{T}_{t_0} v_0 = v_0 \]  
Moreover, 
\[ \widehat{T}_{t} v_0 = v_0 \text{ if and only if } t = nt_0 \text{ for some positive integer n} \]

**Proof:** Let \( \omega \) be an arbitrary continuous function such that \(|\omega(x)| \leq Cx^\gamma\), where \( C > 0 \) and \( x \in [0,1] \). Let \( \omega \) satisfying the following conditions

1. \( e^{\mathcal{M}_0} \omega(e^{-t_0}) = \omega(1) \)
2. \( e^{\mathcal{M}} \omega(e^{-t}) \neq \omega(1) \quad \forall t \in (0,1) \)

The function \( \omega \), defined in such way, belong to \( \hat{L}_p \) space because

\[
S_{[0,a]}(\omega) = \sup_{x \in [0,a]} \left[ \frac{1}{x^p} \int_0^x |\omega(s)|^p ds \right] \leq C \sup_{x \in [0,a]} \left[ \frac{1}{x^p} \int_0^x x^\gamma ds \right] \leq C \frac{a^\gamma}{(\gamma p+1)^{1/p}}
\]

This leads to \( \lim_{a \to 0} S_{[0,a]}(\omega) = 0 \), as \( \gamma > 0 \). Since

\[
[0,1] = \bigcup_{n=0}^{\infty} \left( e^{-(n+1)t_0}, e^{-nt_0} \right]
\]
we can define a function \( v \) on the interval \([0,1]\) by squeezing the graph of the function \( \omega \) into intervals \( \left( e^{-(n+1)t_0}, e^{-nt_0} \right) \), we put

\[
v(x) = e^{-nt_0} \omega(xe^{nt_0}) \text{ for } x \in \left( e^{-(n+1)t_0}, e^{-nt_0} \right]
\]

for such function we have

\[
|v(x)| \leq e^{-n\mathcal{M}} \omega(xe^{nt_0}) \leq e^{-n\mathcal{M}} C(xe^{nt_0})^\gamma = Cx^\gamma
\]

So, \( v(0) = 0 \) and we obtain the continuous function defined on the whole interval \([0,1]\). The properties (1) and (2) follows from (3) and (4), respectively as we
above the function which fulfills the condition \( |\psi(x)| \leq Cx^\gamma \) for \( x \in [0,1] \) belong to \( \hat{L}_p \) space. This finish the proof.

**Theorem 4.2:** If \( \gamma > 0 \) the set of periodic points of \( (1) \) is dense in \( \hat{L}_p \) \( p < 1 \).

**Proof:** Let \( \omega \in \hat{L}_p \) be an arbitrary function. Define a new function

\[
\omega_c(x) = \begin{cases} 
\omega(x) & \text{for } |\omega(x)| \leq Cx^\gamma \\
\text{sgn}(\omega(x)) \cdot Cx^\gamma & \text{for } |\omega(x)| > Cx^\gamma 
\end{cases}
\]

where \( C > 0 \). It is obvious that \( \omega_c \in \hat{L}_p \). At first we claim that for such that functions

\[
\lim_{c \to \infty} S_{e^{-t_c},1} (\omega_c - \omega) = 0
\]

\[
S_{e^{-t_c},1} (\omega_c - \omega) = \sup_{x \in [e^{-t_c},1]} \left( \frac{1}{x} \int_0^x [\omega_c(s)-\omega(s)]^p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \sup_{x \in [e^{-t_c},1]} \left( \frac{1}{x} \int_{s \in [0,1]} |\omega(s)|^p \, ds \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{e^{-t_c}} \int_{s \in [0,1]} |\omega(s)|^p \, ds
\]

We know that for arbitrary \( \eta > 0 \)
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\[
\left( \int_0^1 |\omega(s)|^p \, ds \right)^{\frac{1}{p}} \geq \left( \int_{s \in [0,1]} : |\omega(s) > c \nu| \, |\omega(s)|^p \, ds \right)^{\frac{1}{p}} \\
\geq \left( \mu \left( \{ s \in [0,1] : \omega(s) > c \nu \} \right) \right)^{\frac{1}{p}} (c \nu)
\]

where \( \mu \) is the measure in \( \mathbb{R} \). We will show that the measure of the interval \( \{ s \in [0,1] : \omega(s) > c \nu \} \) tend to zero where \( c \to \infty \). Using the above estimation we have for any \( \eta > 0 \)

\[
\frac{1}{c \nu} \left( \int_0^1 |\omega(s)|^p \, ds \right)^{\frac{1}{p}} \geq \left( \mu \left( \{ s \in [0,1] : \omega(s) > c \nu \} \right) \right)^{\frac{1}{p}} \\
\mu \left( \{ s \in [0,1] : \omega(s) > c \nu \} \right) \leq \eta + \mu \left( \{ s \in [0,1] : s > \eta \land \omega(s) > c \nu \} \right)
\]

\[
\leq \eta + \frac{1}{c \nu} \left( \int_0^1 |\omega(s)|^p \, ds \right)^{\frac{1}{p}}
\]

Since \( \eta \) is arbitrary, it follows that \( \mu \left( \{ s \in [0,1] : \omega(s) > c \nu \} \right) \to 0 \) when \( c \to \infty \).

This completes our claim. Fix \( t_0 \) and constant \( C \) such that

\[
S_{e^{-t_0} \cdot 1} (\omega_c(s) - \omega(s)) \leq e^{\frac{1}{C}} \left( \int_{s \in [0,1]} : |\omega(s) > c \nu| \, |\omega(s)|^p \, ds \right)^{\frac{1}{p}}
\]

We are going to define periodic point by formula

\[
\nu(x) = e^{-n \gamma}, \omega_c (xe^{nt_0}), x \in \left[ e^{-(n+1)}, e^{-nt_0} \right]
\]

for a good Choice of the parameter \( t_0 \). Both function \( \omega_c \) and \( \nu \) belong to \( \hat{L}_p \) so
\[ e^p \int_{[0,e^{-t}]} |v(s) - \omega_c(s)|^p \, ds \leq \frac{\varepsilon}{4} \]

Finally,

\[ S_{[0,1]} (v-\omega) \leq S_{[0,e^{-t}]} (v-\omega) + S_{[e^{-t},1]} (v-\omega) \]

\[ \leq S_{[0,e^{-t}]} (v) + S_{[0,e^{-t}]} (\omega) + S_{[e^{-t},1]} (v-\omega_c) + S_{[e^{-t},1]} (\omega_c - \omega) < \varepsilon \]

**Theorem 4.3:** If \( \gamma \leq 0 \), then the semi group \( (\tilde{T}_t)_{t \geq 0} \) is strongly stable, if \( \gamma < 0 \), \( (\tilde{T}_t)_{t \geq 0} \) is exponentially stable.

**Proof:** Let \( v \in \hat{L}_p \) be an arbitrary function.
\[ S_P^{(0,1)}(\tilde{T}_t^x) = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x |\tilde{T}_t^x v(s)|^p ds \right) \]

\[ = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x e^{\gamma t} v(se^{-t}) |s|^p ds \right) = e^{(p+1)t} \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x v(s)|^p ds \right) \]

\[ = e^{\gamma t} \sup_{x \in (0,1)} \left( \frac{1}{xe^{-t}} \int_0^{xe^{-t}} |v(s)|^p ds \right) = e^{\gamma t} \sup_{x \in (0,e^{-t})} \left( \frac{1}{x} \int_0^x v(s)|^p ds \right) \]

\[ 0 < p < 1, \gamma \leq 0 \Rightarrow e^{\gamma t} \leq 1 \]

Since \( e^{\gamma t} \leq 1 \) by definition \( S_{(0,e^{-t})}(v) \to 0 \) as \( t \to \infty \) we obtain strong stability of the system \( (\tilde{T}_t^x)_{t \geq 0} \in \hat{L}_p \).

The second part of proof follows from the above inequality, too we gain exponential stability with \( D = 1 \) and \( \omega = -\gamma \)

\[ \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x |\tilde{T}_t^x v(s) ds|^{p \over 2} \right) = \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x e^{\tau t} v(se^{-t})|s|^p ds \right)^{\frac{1}{p}} = \]

\[ e^{\gamma t} \sup_{x \in (0,1)} \left( \frac{1}{x} \int_0^x |v(s)|^p ds \right)^{\frac{1}{p}} \leq De^{-\omega t} = e^{-(\gamma) t} = e^{\gamma t} \]

References


Received: March 15, 2013