Some Properties of Class of $p$-Supremum Bounded Variation Sequences

Moch Aruman Imron

Department of Math, Faculty of Math and Sciences
University of Brawijaya, Malang and
Graduate School, Faculty of Mathematics and Sciences
University of Gadjah Mada, Yogyakarta, Indonesia

Ch. Rini Indrati and Widodo

Department of Math, Faculty of Math and Sciences
University of Gadjah Mada, Yogyakarta, Indonesia

Abstract

In this paper, we investigate some properties of class of $p$-Supremum Bounded Variation Sequences. It is a generalization of Supremum Bounded Variation Sequences. The properties will be used to investigate uniform convergence of trigonometric series with $p$-Supremum Bounded Variation Sequences coefficients.

Keywords: $p$-Supremum Bounded Variation Sequences, uniform convergence, trigonometric series

1. Introduction

In Fourier analysis, it is well known that there are a great number of interesting results established by assuming monotonicity of coefficients. The following classical result in Theorem 1.1. was proved by Chaundy and Jollife [9]:

Theorem 1.1. Suppose that $\{a_k\} \subset (0,\infty)$ is nonincreasingly tending to zero.
The series
\[ \sum_{k=1}^{\infty} a_k \sin kx \]  
converges uniformly in \( x \) if only if \( \lim_{k \to \infty} k a_k = 0 \).

Theorem 1.1. has been generalized by many authors, to weaken the monotone conditions of the coefficients sequence in (1.1), by introducing classes GMS (General Monotone Sequences) [8], MVBVS (Mean Value Bounded Variation Sequences) [7], SBVS (Supremum Bounded Variation Sequences), and SBVS2 [5]. Here are the definition of them.

**Definition 1.2.** A sequence \( \{a_k\} \subset [0, \infty) \) is said to be in class

(i) GMS (General Monotone Sequences) if there exists a positive constant \( C \), depending only on \( \{a_k\} \), such that
\[
\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C a_n, \quad n \geq 1.
\]

(ii) MVBVS (Mean Value Bounded Variation Sequences) if there exist positive constant \( C \) and \( \lambda \geq 2 \) depending only on \( \{a_k\} \), such that
\[
\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C \sum_{k=n}^{\lfloor \lambda n \rfloor} |a_k|, \quad n \geq 1,
\]

where \( \lfloor x \rfloor \) denotes the greatest integer that is less than or equal to \( x \).

(iii) SBVS (Supremum Bounded Variation Sequences) if there exist positive constant \( C \) and \( \gamma \), depending only on \( \{a_k\} \), such that
\[
\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C \sup_{m \leq n} |a_m| \sum_{k=m}^{2m} |a_k|, \quad n \geq 1.
\]

(iv) SBVS2 if there exist positive constant \( C \) and \( \gamma \) tending monotonically to infinity, depending only on \( \{a_k\} \), such that
\[
\sum_{k=n}^{2n-1} |a_k - a_{k+1}| \leq C \sup_{m \leq n} |a_m| \sum_{k=m}^{2m} |a_k|, \quad n \geq 1.
\]

It has been shown that \( GMS \subseteq MVBVS \subseteq SBVS \subseteq SBVS2 \) [6, 8]. Further, Liflyand and Tikhonov [1, 2] defined a class of \( p \)-General Monotone Sequences (\( \mathcal{G}MS_p \)) as follows:

**Definition 1.3.** Let \( a = \{a_n\} \) and \( \beta = \{\beta_n\} \) be two sequences of complex and positive numbers, respectively. The couple \((a, \beta)\) determines a \( p \)-General Monotone Sequences, written \((a, \beta) \in \mathcal{G}MS_p\), if there exists a positive constant \( C \) and the relation
\[
(\sum_{k=n}^{2n-1} |a_k - a_{k+1}|^p)^{1/p} \leq C \beta_n, \quad n \geq 1,
\]
holds for \( p, \ 1 \leq p < \infty \).

Let \( a = \{a_n\} \) be a sequence of complex number and \( \beta = \{\beta_n\} \), where \( \beta_n = |a_n| \cdot (a, \beta) \in \mathcal{G}MS_1 \) if and only if \( a \in \mathcal{G}MS \) [2]. As corollary, \( \mathcal{G}MS_p \) is more general than \( \mathcal{G}MS \). Futhermore, Imron, et. al. [5, 6] generalized MVBVS and SBVS to \( \mathcal{M}VBVS_p \) and \( \mathcal{SBVS}_p \), respectively, as stated in Definition 1.4. These generalization results give an opportunity to search the uniform convergence in a bigger class.
Definition 1.4. Let $a = \{a_n\}$ and $\beta = \{\beta_n\}$ be two sequences of complex and positive numbers, respectively. For $1 \leq p < \infty$, a couple $(a, \beta)$ is said to be

(i) $p$-MeanValue Bounded variation Sequences, written $(a, \beta) \in \mathcal{MVBVS}_p$, if there exist a positive constant $C$ and $\lambda \geq 2$ such that

$$
\left(\sum_{k=n}^{2n} |a_k - a_{k+1}|^{1/p}\right)^{1/p} \leq C \sum_{k=\lfloor n/\lambda \rfloor}^{\lfloor n/\lambda \rfloor + 1} \beta_k, \quad n \geq 1,
$$

(ii) $p$-Supremum Bounded Variation Sequences, written $(a, \beta) \in \mathcal{SBS}_p$, if there exist positive constant $C$ and $\gamma \geq 1$ such that

$$
\left(\sum_{k=n}^{2n} |a_k - a_{k+1}|^{1/p}\right)^{1/p} \leq \frac{C}{n} \left(\sup_{m \geq \gamma(n)} \sum_{k=m}^{2m} \beta_k\right), \quad n \geq 1.
$$

The class of $\mathcal{SBS}_p$ has been generalized to be $\mathcal{SBS}_2^p$, $1 \leq p < \infty$, as given in Definition 1.5.

Definition 1.5. Let $a = \{a_n\}$ and $\beta = \{\beta_n\}$ be two sequences of complex and positive numbers, respectively. For $1 \leq p < \infty$, a couple $(a, \beta)$ is said to be a $p$-Supremum Bounded Variation Sequences, written $(a, \beta) \in \mathcal{SBS}_p$, if there exist a positive constant $C$ and $\{b_k\} \subseteq [0, \infty)$ tending monotonically to infinity depending only on $\{a_k\}$, such that

$$
\left(\sum_{k=n}^{2n} |a_k - a_{k+1}|^{1/p}\right)^{1/p} \leq \frac{C}{n} \left(\sup_{m \geq \gamma(n)} \sum_{k=m}^{2m} \beta_k\right), \quad n \geq 1.
$$

Imron, et. al. [5] have shown that $\mathcal{MVBVS}_p \subseteq \mathcal{SBS}_p \subseteq \mathcal{SBS}_2^p$, however, it has not proved the inclusion is inclusive or exclusive. In this present paper, we prove that the inclusion is exclusive, i.e., $\mathcal{MVBVS}_p \not\subseteq \mathcal{SBS}_p$ (Theorem 2.1) and $\mathcal{SBS}_p(\beta) \not\subseteq \mathcal{SBS}_2^p(\beta)$ (Theorem 2.4).

In this paper, we also generalize some properties of $p$-Supremum Bounded Variation Sequences class. We use the properties to investigate the uniform convergence of the sine and cosine series.

2. Main Result

In this section, we investigate the proper subset relations between class of $p$-Mean Value Bounded Variation Sequences and class of $p$-Supremum Bounded Variation Sequences. Furthermore, we study the other properties of class of $p$-Supremum Bounded Variation Sequences.

Theorem 2.1. If $1 \leq p < \infty$, then $\mathcal{MVBVS}_p \not\subseteq \mathcal{SBS}_p$.

Proof: By Theorem 3.8. [4], it has shown that $\mathcal{MVBVS}_p \subseteq \mathcal{SBS}_p$. Now let us show an example a couple $(a, \beta) \in \mathcal{SBS}_p$ but $(a, \beta) \not\in \mathcal{MVBVS}_p$. For $j = 1, 2, 3, ...$, set $n_j = 2^{2^j}$ and
We define the sequence \( \{ \beta_k \} \), where \( \beta_k = a_k \), for every \( k \).

Moreover, for \( j \geq 2 \) such that \( n_{j-1} \leq n \leq n_j \)

\[
\left( \sum_{k=n_{j-1}}^{2n_{j-1}} |\Delta a_k|^p \right)^{1/p} \leq 2 = \frac{2}{n_j} \sum_{k=n_j}^{2n_j} \beta_k \leq \frac{2}{n} \sup_{m \geq n} \sum_{k=m}^{2m} \beta_k
\]

and for \( n_j^2 \leq n \leq 2n_j^2 \)

\[
\left( \sum_{k=n_{j-1}}^{2n_{j-1}} |\Delta a_k|^p \right)^{1/p} \leq 2 = \frac{2}{n_j} \sum_{k=n_j}^{2n_j} \beta_k \leq \frac{4}{n} \sup_{m \geq [n/2]} \sum_{k=m}^{2m} \beta_k.
\]

For \( 1 \leq n < n_1 \)

\[
\left( \sum_{k=n_{j-1}}^{2n_{j-1}} |\Delta a_k|^p \right)^{1/p} = 0 \leq \frac{2}{n} \sup_{m \geq n} \sum_{k=m}^{2m} \beta_k.
\]

Therefore, \( (a, \beta) \in SBVS_p \) by \( C = 4 \) and \( \gamma = 2 \).

On the other hand, we have

\[
\sum_{k=n_j}^{2n_j-1} |\Delta a_k| = 2
\]

and

\[
\left( \sum_{k=n_j}^{2n_j-1} |\Delta a_k|^p \right)^{1/p} = 2.
\]

As corollary,

\[
\frac{C}{n_j} \sum_{j=1}^{\lambda n_j} |\Delta a_k| = \frac{C}{n_j} 2 < \left( \sum_{k=n_j}^{2n_j-1} |\Delta a_k|^p \right)^{1/p}
\]

for all \( C > 0 \), \( n_j > C \), and \( \lambda \geq 2 \). This leads to a contradiction. Therefore, \( (a, \beta) \notin MVBVS_p \). That means, \( MVBVS_p \nsubseteq SBVS_p \). \( \square \)

**Definition 2.2.** Let \( \beta \) be a sequence of positive numbers.

(i) A class of p-Mean Value Bounded Variation Sequences of \( \beta \), written \( MVBVS_p(\beta) \), is defined as

\[
\{ a : (a, \beta) \in MVBVS_p \}.
\]

(ii) A class of p-Supremum Bounded Variation Sequences of \( \beta \), written \( SBVS_p(\beta) \), is defined as

\[
\{ a : (a, \beta) \in SBVS_p \}.
\]

Definition 2.2 and Theorem 2.1 implies corollary 2.3.

**Corollary 2.3.** If \( 1 \leq p < \infty \), then \( MVBVS_p(\beta) \nsubseteq SBVS_p(\beta) \).
**Theorem 2.4.** If \( 1 \leq p < \infty \), then \( SBVS_p \subseteq SBVS2_p \).

**Proof.** By Theorem 3.5. [4], it has been shown that \( SBVS_p \subseteq SBVS2_p \). Now let us show an example for \( (a, \beta) \in SBVS2_p \) and \( (a, \beta) \notin SBVS_p \). For \( j = 1, 2, 3, ... \), set \( n_j = 2^{2^j} \) and

\[
a_k = \begin{cases} 
0 & \text{if } 1 \leq k \leq n_1 - 1, \\
n_j^{-2} & \text{if } k = n_j \\
n_j^{-3/2} & \text{if } n_j < k < n_j^2, \\
n_j^{-1} & \text{if } n_j^2 \leq k \leq 2n_j^2 \\
0 & \text{if } 2n_j^2 < k < n_{j+1}.
\end{cases}
\]

We define the sequence \( \{\beta_k\} \), where \( \beta_k = a_k \), for every \( k \). For \( n, n_{j-1} \leq n \leq n_j \) and \( j \geq 2 \) we have

\[
\left( \sum_{k=n}^{2n_j-1} |\Delta a_k|^p \right)^{1/p} \leq \frac{2}{n_j} \sum_{k=n_j}^{2n_j^2-1} a_k \leq \frac{2}{n_j^2} \sum_{k=n_j}^{2n_j^2} a_k \leq \frac{2}{n} \sup_{m \geq n^{1/2}} \sum_{k=m}^{2m} \beta_k.
\]

For \( n, 1 \leq n < n_1 \) we have

\[
(\sum_{k=n}^{2n-1} |\Delta a_k|^p)^{1/p} \leq \frac{2}{n^2} \leq \frac{2}{n} \sup_{m \geq n^{1/2}} \sum_{k=m}^{2m} \beta_k.
\]

So \( (a, \beta) \in SBVS2_p \) with \( C = 2 \) and \( b_n = n^{1/2} \).

On the other hand, we can show that \( (a, \beta) \notin SBVS_p \). To see this, calculate summation

\[
\sum_{k=n_j}^{2n_j-1} |\Delta a_k| = \frac{1}{n_j^2}
\]

and

\[
\left( \sum_{k=n_j}^{2n_j-1} |\Delta a_k|^p \right)^{1/p} = \frac{1}{n_j^2}.
\]

Then

\[
\frac{C}{n_j} \sup_{m \geq \gamma} \sum_{k=m}^{2m} |\beta_k| = \frac{C}{n_j} \max \left( \sup_{k \geq j} \frac{1}{n_k^2}, \sup_{k > j} \frac{1}{n_k^{3/2}} \right) = \frac{C}{n_j^3} \frac{1}{n_j^3} \leq \frac{1}{n_j^3}
\]

for all \( C > 0 \), \( \gamma \geq 1 \) and \( n_j > C \). This leads to a contradiction. Therefore, \( (a, \beta) \notin SBVS_p \) then \( SBVS_p \subsetneq SBVS2_p \). \( \blacksquare \)
Definition 2.5. Let \( \beta \) be a sequence of positive numbers and \( 1 \leq p < \infty \). A class of \( p \)-Mean Value Bounded Variation Sequences two of \( \beta \), written \( SBVS_p(\beta) \), is defined as \( \{ a: (a, \beta) \in SBVS_p \} \).

By Theorem 2.4 and Definition 2.5, we have the following corollary.

Corollary 2.6. If \( 1 \leq p < \infty \), then \( SBVS_p(\beta) \subset SBVS_2(\beta) \).

Some properties of \( SBVS_2 \), \( 1 \leq p < \infty \), are stated below. Theorem 2.7 and Theorem 2.9 are the generalization of Lemma 3.1 and Lemma 3.4 in [8], respectively.

Theorem 2.7. Let \( 1 \leq p < \infty \). If \( (a, \beta) \in SBVS_2 \), then for any integer \( n \)

(i). \( |a_v| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_m| \) for any \( v, m = n, \ldots, 2n \).

(ii). \( |a_v| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + \frac{1}{n} \sum_{j=n+1}^{2n} |a_j| \) for any \( v = n, \ldots, 2n \).

Proof.

(i). For each \( k, a_k \) can be stated as follows:

\[
\begin{align*}
  a_k &= (a_k - a_{k+1}) + \cdots + (a_{j-1} - a_j) + a_j = \sum_{s=k}^{j-1} \Delta a_s + a_j \quad \text{and} \\
  a_k &= -(a_i - a_{i+1}) - \cdots - (a_{k-1} - a_k) + a_i = -\sum_{s=i}^{k-1} \Delta a_s + a_i,
\end{align*}
\]

Therefore

\[
|a_k| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_m| \]

By using the Holder’s inequality, we have

\[
|a_k| \leq \left( \sum_{s=n}^{2n-1} |\Delta a_s|^p \right)^{1/p} \left( \sum_{s=n}^{2n-1} 1^{1-1/p} \right)^{1-1/p} + |a_m|. \]

Since \( (a, \beta) \in SBVS_2 \), there exist positive constant \( C \) and \( \{ b_r \} \subset [0, \infty), b_r \to \infty \) as \( r \to \infty \), such that

\[
|a_k| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_m|
\]

for any \( k, m = n, \ldots, 2n \).

(ii). According to Theorem 2.1.i, we have

\[
|a_k| \leq \left( \sum_{s=n}^{2n-1} |\Delta a_s|^p \right)^{1/p} \left( \sum_{s=n}^{2n-1} 1^{1-1/p} \right)^{1-1/p} + |a_m|
\]

\[
\leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_m|
\]

for any \( k, m = n, \ldots, 2n \). Suppose \( m = k + 1, \ldots, 2n \), then

\[
|a_k| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_{2n}|
\]

\[
|a_k| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_{2n-1}|
\]

\[
\]
Corollary 2.8. Let $1 \leq p < \infty$. If $(a, \beta) \in \text{SBV}S_p$, then for any $n$ and $\gamma \geq 1$, we have

(i). $|a_v| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + |a_m| \; \text{for any } v, m = n, \ldots, 2n.$

(ii). $|a_v| \leq C n^{-1/p} \left( \sup_{j \geq b_n} \sum_{s=j}^{2j} \beta_s \right) + \frac{1}{n} \sum_{j=n+1}^{2n} |a_j| \; \text{for any } v = n, \ldots, 2n.$

Theorem 2.9 give a sufficient condition for a couple of sequences in $\text{SBV}S_2$ to be have bounded variation.

Theorem 2.9. Let $1 \leq p < \infty$. If $(a, \beta) \in \text{SBV}S_2$, and

$$\sum_{v=[n/2]}^{\infty} v^{-1-1/p} \left( \sup_{j \geq b_v} \sum_{s=j}^{2j} \beta_s \right) < \infty,$$

then $a = \{a_n\}$ is bounded variation. Moreover, for each $n \in \mathbb{N}$, $N \geq n$ we have

$$\sum_{v=[n/2]}^{N} |\Delta a_v| \leq C \sum_{v=[n/2]}^{2N} v^{-1-1/p} \left( \sup_{j \geq b_v} \sum_{s=j}^{2j} \beta_s \right).$$

Proof. Suppose $[n/2] \geq 1$. We have

$$\sum_{k=n}^{N} |\Delta a_k| = |\Delta a_n| + |\Delta a_{n+1}| + \cdots + |\Delta a_{N}|$$

$$\leq N \left( \frac{1}{[n/2]} \left( |\Delta a_{[n/2]}| + |\Delta a_{[n/2]+1}| + \cdots + |\Delta a_{2[n/2]-1}| \right) + \frac{1}{[n/2]} \left( |\Delta a_{[n/2]+1}| + |\Delta a_{[n/2]+2}| + \cdots + |\Delta a_{n}| + \cdots + |\Delta a_{2[n/2]+1}| \right) \right.$$

$$\left. + \frac{1}{2N} \left( |\Delta a_{2N}| + |\Delta a_{2N+1}| + \cdots + |\Delta a_{4N-1}| \right) \right)$$

$$= N \sum_{j=[n/2]}^{2N} \frac{1}{j} \sum_{k=j}^{2j-1} |\Delta a_k|.$$

By Holder’s inequality,

$$\sum_{k=j}^{2j-1} |\Delta a_k| \leq \left( \sum_{k=j}^{2j-1} |\Delta a_k|^p \right)^{1/p} \left( (2j-1) - (j-1) \right)^{1-1/p}.$$
Since \((a, \beta) \in \mathcal{SBVS}_2\), there exist positive constant \(C'\) and \(\{b_r\} \subset [0, \infty)\), \(b_r \to \infty\) as \(r \to \infty\), such that
\[
\left( \sum_{k=j}^{2^j-1} |\Delta a_k|^p \right)^{\frac{1}{p}} \leq \frac{C'}{j} \left( \sup_{\nu \geq b_j} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right).
\] (2.4)

From (2.3) and (2.4), we have
\[
\sum_{k=n}^{N} |\Delta a_k| \leq N \sum_{j=\lceil n/2 \rceil}^{2N} \frac{1}{j} \sum_{k=j}^{2^j-1} |\Delta a_k|
\leq NC' \sum_{j=\lceil n/2 \rceil}^{2N} j^{-1-1/p} \left( \sup_{\nu \geq b_j} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right)
= C \sum_{j=\lceil n/2 \rceil}^{2N} \left( j^{-1-1/p} \sup_{\nu \geq b_j} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right),
\]
where \(C = NC'\).

Since \(C \sum_{j=\lceil n/2 \rceil}^{2N} \left( j^{-1-1/p} \sup_{\nu \geq b_j} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right) < \infty\), then \(\sum_{k=n}^{\infty} |\Delta a_k| < \infty\).
Therefore \(a = \{a_n\}\) is bounded variation. ■

**Corollary 2.10.** Let \(1 \leq p < \infty\). If \((a, \beta) \in \mathcal{SBVS}_p\), \(\gamma \geq 1\) and \(\sum_{\nu = \lceil n/2 \rceil}^{\infty} \nu^{-1-1/p} \left( \sup_{\nu \geq \nu} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right) < \infty\), then \(a = \{a_n\}\) is bounded variation. Moreover,
\[
\sum_{\nu = n/2}^{\infty} |\Delta a_{\nu}| \leq C \sum_{\nu = \lceil n/2 \rceil}^{2N} \left( \nu^{-1-1/p} \left( \sup_{\nu \geq \nu} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right) \right).
\]

### 3. Uniform Convergence of Sine and Cosine Series

Dyachenko and Tikhonov \[3\] discuss the uniform convergence of class of \(GMS_1\) to sine and cosine series. In this section, we discuss the uniform convergence of class of \(\mathcal{SBVS}_2\), \(1 \leq p < \infty\).

We consider the series
\[
\sum_{k=1}^{\infty} a_k \cos kx
\] (3.1)
and
\[
\sum_{k=1}^{\infty} a_k \sin kx
\] (3.2)
where \(a = \{a_k\}\) is a given null sequence of complex numbers, i.e., \(a_k \to 0\) as \(k \to \infty\). We define by \(f\) and \(g\) the sums of series (3.1) and (3.2), respectively, at the point where the series converge.

**Theorem 3.1.** Let \((a, \beta) \in \mathcal{SBVS}_2\), \(1 \leq p < \infty\). If
\[
n \sum_{\nu = \lceil n/2 \rceil}^{\infty} \nu^{-1-1/p} \left( \sup_{\nu \geq b_\nu} \sum_{\nu = \nu}^{2\nu} \beta_{\nu} \right) = o(1),
\]
the series (3.2) converges uniformly on \([0, 2\pi]\).
Class of p-supremum bounded variation sequences

Proof.
(i). For $x \neq 0$. We denote

$$d_n = n \sum_{p=\lfloor n/2 \rfloor}^{\infty} v^{-1/p} \left( \sup_{j \geq b_p} \sum_{s=j}^{2j} \beta_s \right).$$

It is clear that $d_n \to 0$. Let $\varepsilon > 0$ be given, then there exists $n_0 \in \mathbb{N}$ such that $|d_n| < \varepsilon$ for $n \geq n_0$. Let $\{v_n\}$ be nonincreasing null sequence such that $|\sum_{j=n_0}^{\infty} a_j| \leq v_n$, so there exists $n_1 \in \mathbb{N}$ such that $|v_n| < \varepsilon$ for $n \geq n_1$. Given $(a, \beta) \in S^\mathcal{B}_2$ and $n \sum_{p=\lfloor n/2 \rfloor}^{\infty} v^{-1/p} \left( \sup_{j \geq b_p} \sum_{s=j}^{2j} \beta_s \right) = o(1)$, by Theorem 2.10

$$n \sum_{p=\lfloor n/2 \rfloor}^{\infty} |\Delta a_p| \leq nC \sum_{p=\lfloor n/2 \rfloor}^{\infty} v^{-1/p} \left( \sup_{j \geq b_p} \sum_{s=j}^{2j} \beta_s \right) \leq C d_n \quad (3.3)$$

Let us now estimate $g(x) = S_n(g, x)$, where

$$S_n(g, x) = \sum_{j=1}^{n} a_n \sin jx.$$  

By Abel’s transformation, we get

$$g(x) - S_{n-1}(g, x) = \sum_{k=n}^{\infty} \Delta a_k D^*_n(x) - a_n D^*_{n-1}(x) = A + B$$

Where $D^*_n(x) = \sum_{j=1}^{n} \sin jx$ and $A = \sum_{k=n}^{\infty} \Delta a_k D^*_n(x)$, $B = -a_n D^*_{n-1}(x)$

By (3.3)

$$|B| = |a_n D^*_{n-1}(x)| \leq n|a_n| \leq n \sum_{p=\lfloor n/2 \rfloor}^{\infty} |\Delta a_p| \leq C d_n.$$  

To estimate $A$, for any $x \in (0, \pi]$ we can find $N \in \mathbb{N}$ such that

$$x \in \left( \frac{\pi}{N+1}, \frac{\pi}{N} \right).$$  

Since $\sum_{p=1}^{n} \sin px = \frac{\cos \frac{1}{2}px - \cos \frac{1}{2}(n+1)x}{2 \sin \frac{1}{2}x} \leq \frac{1}{\sin \frac{1}{2}x} \leq \frac{2}{\pi x}$, if $N \leq n$, then

$$D^*_n(x) = \sum_{p=1}^{n} \sin px \leq \frac{2}{\pi x} \quad \text{and (3.3) imply}$$

$$|A| \leq \frac{C}{x} \sum_{k=n}^{\infty} |\Delta a_k| \leq CN \frac{d_n}{n} \leq C d_n \quad (3.4)$$

If $N > n$, then we decompose $A$ as $A = \sum_{k=n}^{N-1} \Delta a_k D^*_n(x) + \sum_{k=N}^{\infty} \Delta a_k D^*_n(x)$. Similar to (3.3) we get

$$|\sum_{k=n}^{N-1} \Delta a_k D^*_n(x)| \leq CN \sum_{k=n}^{\infty} |\Delta a_k| \leq C d_N \leq C d_n.$$  

Further

$$|\sum_{k=n}^{N-1} \Delta a_k D^*_n(x)| \leq |\sum_{k=n}^{N-1} \Delta a_k| \leq |\sum_{k=n}^{N-1} \Delta a_k||D^*_n(x) - k| = L + M$$

where $L = |\sum_{k=n}^{N-1} \Delta a_k|$ and $M = |\sum_{k=n}^{N-1} \Delta a_k||D^*_n(x) - k|$. Since

$$|\sum_{k=n}^{N-1} \Delta a_k| = \sum_{k=n}^{N-1} a_k + (n-1)a_n - (N-1)a_N \leq \sum_{k=n}^{N-1} a_k + a_N + n \sum_{k=n}^{\infty} \Delta a_k$$

then $L \leq 2v_n + Cd_n$.

From $|D^*_n(x) - k| \leq k^2x$, we obtain

$$M \leq x \sum_{k=n}^{N-1} k^2 |\Delta a_k| = x(n^2|\Delta a_n| + (n+1)^2|\Delta a_{n+1}| + \cdots + (N-1)^2|\Delta a_{N-1}|) + x(n^2|\Delta a_n| + (n^2 + 2n + 1)|\Delta a_{n+1}| + \cdots + (N-1)^2|\Delta a_{N-1}|)$$

$$= x(n^2|\Delta a_n| + (n^2 + 2n + 1)|\Delta a_{n+1}| + \cdots + (N-1)^2|\Delta a_{N-1}|)$$
\[ \leq \frac{C}{\sum_{j=n}^{N} |\Delta a_j| + n \sum_{j=n}^{N} |\Delta a_j| + (n + 1) \sum_{j=n+1}^{N} |\Delta a_j| + \cdots + N|\Delta a_{N-1}|) }{\sum_{j=n}^{N} |\Delta a_j| + \sum_{m=n}^{N} m \sum_{j=m}^{N} |\Delta a_j|} \leq \frac{C}{\sum_{j=n}^{N} (n \Delta a_j + \sum_{m=n}^{N} a_m)} \leq C d_n. \] (3.5)

From (3.3), (3.4) and (3.5), we get
\[ |g(x) - S_n(g, x)| \leq A + B \leq C (d_n + n^2) \]
So, if given \( \varepsilon > 0 \) there exists \( n_2 = \max(n_0, n_1) \) such that for \( n \geq n_2 \)
\[ |g(x) - S_n(g, x)| \leq A + B \leq 2C \varepsilon \]
The series (3.2) converges uniformly on \((0, \pi]\).
(ii). For \( x = 0 \), \( \sum_{k=1}^{\infty} a_k = 0 \) and from i, then (3.2) converges uniformly on \([0, 2\pi]\).
The proof is complete. \( \blacksquare \)

The uniform convergence of the series in (3.1) in class of \( SBVS2_p \), \( 1 \leq p < \infty \), is stated in Theorem 3.2. We abandon the proof, since it is similar to the proof of (i) in Theorem 3.1.

**Theorem 3.2.** Let \((a, \beta) \in SBVS2_p, 1 \leq p < \infty \), if
\[ n \sum_{v=\lfloor n/2 \rfloor}^{\infty} b_v \sum_{j=1}^{2^j} \beta_v = o(1), \]
then the series (3.1) converges uniformly on \((0, 2\pi]\).

**Acknowledgements**

The authors gratefully acknowledge the support of the Department of Mathematics, Faculty of Mathematics and Sciences University of Brawijaya and the Graduate School Department of Mathematics, Faculty of Mathematics and Sciences, University of Gadjah Mada.

**References**


Received: April 19, 2013