Shape Analysis of Generalized Log-Aesthetic Curves

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Abstract
This paper elucidates the properties of Generalized Log-Aesthetic Curves (GLAC) for CAD practicalities. It is an extension of the emerging Log-Aesthetic (LA) curve with an extra degree of freedom ($\nu$) encompassing clothoid, circle involute, logarithmic spiral and etc. This paper analyzes the bounds of GLAC with respect to its arc length and turning angle. Inflection points occurrence details based on the shape parameters ($\nu$, $\Lambda$, and $\alpha$) are highlighted in this paper.

Mathematics Subject Classification: 65D17, 68U07

Keywords: Log-Aesthetic curves, high quality/fair curves, curvature controlled curves, spirals

1 A Review on Aesthetic Shape Design

In 1999, Harada et al. presented a novel method to investigate curves used in automobile design called Logarithmic Distribution Diagram of Curvature (LDDC) [17]. It is a graph plotted similar to curvature profile, but with $x$-axis representing radius of curvatures and $y$-axis representing its corresponding length frequencies of a segmented curve.
Later, Kanaya et al. represented LDDC analytically and renamed it as $K$-
Vector which rooted from Kansei engineering [4]. It is a Japanese word which
means customer’s physical and psychological emotional responses towards a
product or service in accord with their properties and characteristics. $K$-
Vector was later renamed as Logarithmic Curvature Graph (LCG) [13]. The
advantage of using LCG as a shape interrogation tool compared to curvature
profile is that, user may visually inspect the continuity up to $G^3$ (torsion
continuity) whereas curvature profile’s ability is restricted to $G^2$ continuity.

A spiral is denoted as a curve having either monotonic increase or decrease
of curvature [3, 2]. A visually pleasing curve is said to be a spiral with linear
LCG [17, 4]. Hence, Miura derived a general formula to represent visually
pleasing curves called Log-Aesthetic (LA) curve which preserves the linearity
of LCG [6]. LA curve has a shape parameter denoted as $\alpha$ which dictates the
type of spiral to be used, e.g. LA curve produces Nielsen’s spirals when $\alpha = 0$,
logarithmic spirals when $\alpha = 1$, circle involutes when $\alpha=2$ and clothoids when
$\alpha = -1$. Since the formulation of LCG is complex (involves third derivatives),
Gobithaasan et. al derived a compact formula to derive the LCG gradient by
using curvature radius of an arbitrary curve [13]. Hence, the LCG for curves
in the form of Cesaro equation can be easily derived using the LCG gradient
formula.

The number of research papers produced since the introduction of LA curve
has been increasing exponentially. In 2006, Yoshida & Saito introduced an
algorithm to interactively control the LA curve [9]. In 2009, Levien & Sequin
stated that the LA curve is the most promising curve for aesthetic design [11].
In the same year Gobithaasan & Miura reported that the LCG gradient of
Generalized Cornu Spiral (GCS) [5] can be expressed in a linear form [13].
They extended the family of LA curves by expressing LCG in a linear form
and denoted the resultant curve as a Generalized Log-Aesthetic Curve (GLAC)
[14].

Recent progress in this niche area include reformulation of 2D and 3D LA
curve in the form of variational principal [7], 3D GLAC [15], analytic represen-
tation of LA curve using Incomplete Gamma function for efficient computation
[16] and the formulation of $G^2$ LA spline for automobile design application [8].

1.1 Organization of the Paper

This paper elucidates the properties of GLAC for CAD practicalities. The next
section reviews the formulation of GLAC. Section 3 discusses bound analysis
and the existence of inflection points which can be used for the development
of interactive GLAC design with three control points. Conclusion and future
work is stated in section 5. The final section presents acknowledgement and
references used in this paper.
2 The Formulation of GLAC

A curvature profile, denoted as $\hat{\kappa}(s)$, for an arbitrary curve can be obtained by plotting parameter $s$ against the corresponding curvature value. Curvature radius is the reciprocal of curvature denoted as $\hat{\rho}(s) = \frac{1}{\hat{\kappa}(s)}$. The total area below the curvature profile indicates the total turning angle or denoted as tangential angle, $\hat{\theta}(s)$. A general formulation of planar curves is shown in eqn.(1). $\{x(0), y(0)\}$ is the starting point of the curve.

$$C(s) = \{x(0) + \int_0^s \cos \left[ \hat{\theta}(0) + \int_0^t \hat{\kappa}(u) \, du \right] \, dt, y(0) + \int_0^s \sin \left[ \hat{\theta}(0) + \int_0^t \hat{\kappa}(u) \, du \right] \, dt\}$$

Curve synthesis is a process of formulating planar curves from a defined curvature profile where $s$ is the arc length parameter. GLAC is derived by using the curve synthesis process where the curvature is designed based on the manipulation of GCS and LA curves [14]. The underlying idea was to produce a curve which has an extra degree of freedom and produces a linear function of LCG gradient. To note, the LCG gradient for LA curves is fixed to $\alpha$. Hence, GLAC is a general formulation to represent aesthetic curves.

The curvature (eqn.(2)) and and its turning angle (eqn.(3)) of GLAC is made up of $\alpha$, $\Lambda$ and $\nu$ which are the shape parameters; these variables can be used to shape a GLAC segment which is arc length parameterized. $s$.

$$\kappa(s) = (\Lambda \alpha s + 1)^{-1 / \alpha} + \nu$$

$$\theta(s) = \theta_0 + \frac{1}{\Lambda(\alpha - 1)} \left( (\Lambda \alpha s + 1)^{\frac{\alpha - 1}{\alpha}} - 1 \right) + \nu s$$

where the shape parameters are $\{\alpha, \Lambda & \nu\}$. As stated before, $\alpha$ dictates the type of spiral and $\{\alpha & \Lambda\}$ are free parameters which will be used to satisfy the constraints posed by designers.

The parametric form of GLAC is obtained by replacing the turning angle stated in eqn. (3) into eqn.(1) where $P_0 = \{x(0), y(0)\}$ is the starting point. The curvature radius, $\rho(s)$, for the case of $\alpha = 0$ is defined similar the LA curve where it becomes the Neilsen’s spiral. The $\rho(s)$ piecewise function is shown in eqn.(4).

$$\rho(s) = \begin{cases} \frac{1}{e^{\Lambda s + \nu}} & \text{if } \alpha = 0 \\ \frac{1}{(\Lambda \alpha s + 1)^{\frac{\alpha - 1}{\alpha}} + \nu} & \text{otherwise} \end{cases}$$
The arc length function is derived using eqn.(4) and is stated in eqn.(5).

\[
\begin{align*}
\gamma (\rho) &= \begin{cases} 
\frac{1}{\Lambda} \log \left[ \frac{1}{\rho^{1-\nu}} \right] & \text{if } \alpha = 0 \\
\frac{1}{\Lambda} \left( (\rho^{-1} - \nu)^{-\alpha} - 1 \right) & \text{otherwise}
\end{cases}
\end{align*}
\]

(5)

Eqn.(6) and (7) are piecewise turning angle function of GLAC in terms of arc length and curvature radius derived by imposing three constraints as in LA curves [9].

\[
\begin{align*}
\theta_{GLAC}(s) &= \begin{cases} 
\frac{1}{\Lambda} \left( 1 - e^{-\Lambda s} \right) + \nu s & \text{if } \alpha = 0 \\
\frac{1}{\Lambda} \log[\Lambda s + 1] + \nu s & \text{if } \alpha = 1 \\
\frac{1}{\Lambda(\alpha-1)} \left( (\Lambda s + 1)^{\frac{\alpha-1}{\alpha}} - 1 \right) + \nu s & \text{otherwise}
\end{cases}
\end{align*}
\]

(6)

\[
\begin{align*}
\theta_{GLAC}(\rho) &= \begin{cases} 
\frac{1}{\Lambda} \left( 1 + \nu - \rho^{-1} - \nu \log[\rho^{-1} - \nu] \right) & \text{if } \alpha = 0 \\
\frac{1}{\Lambda} \left( \log[\rho^{-1} - \nu]^{-1} + \nu \left( (\rho^{-1} - \nu)^{-1} - 1 \right) \right) & \text{if } \alpha = 1 \\
\frac{1}{\Lambda(\alpha-1)} \left( (\rho^{-1} - \nu)^{1-\alpha} - 1 \right) + \frac{\nu}{\Lambda} \left( (\rho^{-1} - \nu)^{-\alpha} - 1 \right) & \text{otherwise}
\end{cases}
\end{align*}
\]

(7)

The general formula of LCG for GLAC is shown below:

\[
\begin{align*}
LCG_{GLAC}(s) &= \{ \log \left[ \frac{1}{\Lambda} \left( (\Lambda s + 1)^{-\frac{1}{\alpha}} + \nu \right) \right], \\
& \log \left[ \frac{1}{\Lambda} \left( (\Lambda s + 1)^{\frac{\nu}{\Lambda}} \right) \right] \}
\end{align*}
\]

(8)

3 The Bounds of GLAC

The GLAC shape depends on the bounds of its arc length \((s)\) and turning angle \((\theta)\). The bounds are similar to LA curves when \(\nu = 0\). Hence, the paper focuses on the cases when \(\nu \neq 0\).

The lower bounds (LB) and upper bounds (UB) for \((s)\) and \((\theta)\) can be determined by finding the limits when \(\rho \to 0\) and \(\rho \to \infty\) using eqn.(5) and (7). There are three distinct cases to obtain the bounds; \{\(\nu > 0\)\}, \{\(\nu < 0\)\} and \{\(\nu = -1\)\} as depicted in Table 1 and Table 2.

3.1 The Overall GLAC Shapes

In general, GLAC produces clothoid \((\alpha = -1)\), Nielsen’s spiral \((\alpha = 0)\), logarithmic spiral \((\alpha = 1)\) and circle involute \((\alpha = 2)\) regardless of any \((\nu)\). By
Table 1: Lower & Upper bounds for GLAC's arc length function ($s$)

<table>
<thead>
<tr>
<th></th>
<th>$\nu &gt; 0$</th>
<th>$\nu &lt; 0$</th>
<th>$\nu = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &lt; 0$</td>
<td>LB: $-\frac{1}{\Lambda \alpha}$</td>
<td>UB: $\frac{(-\nu)^{-\alpha} - 1}{\Lambda \alpha}$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$-\frac{1}{\Lambda \alpha}$</td>
<td>$-\frac{\log(-\nu)}{\Lambda \alpha}$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha &gt; 0$</td>
<td>$-\frac{1}{\Lambda \alpha}$</td>
<td>$\frac{(-\nu)^{-\alpha} - 1}{\Lambda \alpha}$</td>
<td>$-\frac{1}{\Lambda \alpha}$</td>
</tr>
</tbody>
</table>

Table 2: Lower & Upper bounds for GLAC's turning angle ($\theta$)

<table>
<thead>
<tr>
<th></th>
<th>$\nu &gt; 0$</th>
<th>$\nu &lt; 0$</th>
<th>$\nu = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha &lt; 0$</td>
<td>LB: $A$</td>
<td>UB: $B$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha = 0$</td>
<td>$-\frac{1}{(1 + \nu - \nu \log(-\nu))}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$0 &lt; \alpha &lt; 1$</td>
<td>$-\frac{1}{(1 + \nu + \log(-\nu))}$</td>
<td>$-\frac{1}{\Lambda \alpha(1-\alpha)}$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha &gt; 0$</td>
<td>$A$</td>
<td>$A$</td>
<td>$B$</td>
</tr>
</tbody>
</table>

where $A = \frac{1}{\Lambda} (\frac{1}{1-\alpha} - \frac{\nu}{\alpha})$ & $B = \frac{1}{\Lambda(\alpha-1)} \left( (-\nu)^{1-\alpha} - 1 \right) + \frac{\nu}{\Lambda \alpha} \left( (-\nu)^{-\alpha} - 1 \right)$.

taking the limit of $\alpha \to \infty$, curvature radius becomes $\rho(s) = 1/(1 + \nu)$. This indicates that GLAC becomes a circle with a radius depending on $\nu$. From eqn.(4) it is understandable that when $\Lambda = 0$, GLAC becomes a circle having $\rho(s) = 1/(1 + \nu)$ for arbitrary value of $\alpha$.

Curvature at the origin in LAC is restricted to 1 whereas the curvature for GLAC at the origin is $1 + \nu$ which indicates that $\nu$ controls the curvature at the origin. Therefore, GLAC comprises of aesthetic curves with $-\infty < \kappa < +\infty$ at the origin.

### 3.2 Inflection Points of GLAC

Identification of inflection points in GLAC enables designer to join GLAC with a line segment with $G^2$ continuity as the curvature for lines are zero. Inflection points in GLAC can be identified by solving $\kappa(s) = 0$. The curvature is inspected for four cases depending on the values of $\alpha$ and $\nu$. Based on Table 3, for cases where $\nu = -1$, inflection points occur at the origin for arbitrary values of $\alpha$. When $\Lambda = 0$, it becomes a circular arc which means the absence of inflection points. An inflection point may occur at a particular value of parameter $s$ for case $\nu = -1$ when $\alpha = 0$ (Neilsen’s spiral). There are many possibilities for an inflection point to exist for cases $\nu \neq -1 (\alpha = 0)$. Figure ?? illustrates curves labeled with arc lengths $s \in \{-1, -0.5, 0, 0.5, 1\}$ indicating
Table 3: The Occurrence of Inflection Points based on $\alpha$ and $\nu$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\nu$</th>
<th>$s$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neq 0$</td>
<td>$\neq -1$</td>
<td>$-\frac{\nu - \alpha - 1}{\lambda \alpha}$</td>
<td>$-\frac{\log(\nu)}{\lambda}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$-1$</td>
<td>$\alpha \lambda s = 0$</td>
<td>$-\lambda s = 0$</td>
</tr>
</tbody>
</table>

Note: $\Lambda \neq 0$ (for cases $\Lambda = 0$; GLAC produces circular arc, hence no inflection point).

Inflection point occurrence. Figure ?? illustrates the possible combination of $\alpha$, $\Lambda$ and $\nu$ to produce inflection points when $s = 5$ for case $\nu \neq -1$ and $\alpha \neq 0$). To note, there are no possibilities for inflection point to occur when $\nu > 0$.

3.3 Using Bounds for GLAC Shape Design

The identification of bounds and inflection points enables designer to form various GLAC shapes with given three control points. These three points make a triangle which is in the form of $G^1$ Hermite data, where the two end corners are the endpoints and the middle corner forming two lines connecting the end points dictates the tangents for the endpoints.

A LA segment has $\alpha$ and $\Lambda$ as shape parameters which can be used to satisfy constraints that occur during design. The designers would be able to dictate the type of spiral to be used by fixing the $\alpha$ value. A suitable value for $\Lambda$ to generate a LA curve with given three control points is obtained by a bisection method [9].

Unlike LA curves, a GLAC segment has three shape parameter that can be used for interactive design. Similar to LA curves, $\alpha$ can be used to dictate the type of spiral, $\Lambda$ to satisfy the $G^1$ Hermite data and $\nu$ to control the curvatures. A GLAC segment can be drawn by searching suitable values for shape parameters ($\Lambda$ and $\nu$) using optimization techniques. Since shape parameter $\nu$ can be used to control the curvatures, hence it is feasible to produce an algorithm satisfying $G^2$ Hermite data with GLAC similar to techniques employed in [12, 1, 8].

4 Conclusion and Future Work

The shape parameters, viz., $\alpha$, $\Lambda$ and $\nu$ gives extra flexibility for the designers to develop desired shapes. This paper identifies GLAC’s bounds and inflection points to facilitate designers’ CAD application. Thus, the designer would be able to create various algorithms to design C-Shape and S-shape GLACs. The
authors are currently working on an algorithm to carry out GLAC that satisfies a two-point $G^1$.

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References


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(a) Case \{\alpha = 0 \text{ and } \nu \neq -1\} \ s \in \{-1, -0.5, 0, 0.5, 1\}

(b) Case \{\alpha \neq 0 \text{ and } \nu \neq -1\} \text{ for } s = 5

Figure 1: The existence of inflection points based on the values of \alpha, \Lambda \text{ and } \nu.