Bipolar ($\lambda, \delta$)-Fuzzy Ideals in Ternary Semigroups

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Abstract

We introduce the concept of bipolar ($\lambda, \delta$)-fuzzification of a ternary semigroup and discuss some structural properties of bipolar ($\lambda, \delta$)-fuzzy ideals of a ternary semigroup.

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1 Introduction

Lehmer [1], studied certain ternary algebraic structures called triplexes in 1932. The notion of ternary semigroup was introduced by Banach. Sioson [2], worked on the ideal theory in ternary semigroups.

A fuzzy subset of a set $S$ is an arbitrary mapping $f : S \rightarrow [0, 1]$, where $[0, 1]$ is the unit segment of a real line. This fundamental concept of fuzzy set was given by Zadeh [3] in 1965. Fuzzy groups have been first considered by Rosenfeld [4] and fuzzy semigroups by Kuroki [5], also see [6, 7].
Lee [8] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$, also see [9]. Bipolar fuzzy sets have been studied in various algebraic structures, see [10, 11, 12, 13, 14].

Yao introduced $(\lambda, \theta)$-fuzzy normal subfields [15]. Khan et al. [16], characterized ordered semigroups in terms of $(\lambda, \theta)$-fuzzy bi-ideals, also see [17].

In this paper, we introduced a notion of bipolar $(\lambda, \delta)$-fuzzy ideals of ternary semigroups and some properties of these ideals are studied.

## 2 Review of Literature

In this section, we will recall some concepts related to ternary semigroups and bipolar fuzzy sets. Throughout the paper $T$ will be considered as a ternary semigroup unless otherwise specified.

**Definition 2.1.** [1] A ternary semigroup $T$ is a non-empty set whose elements are closed under the ternary operation of multiplication and satisfy the associative law defined as

$$[(abc) de] = [a (bcd) e] = [ab [cde]], \quad \text{for all } a, b, c, d, e \in T.$$ 

For simplicity, we shall write $[abc]$ as $abc$. A non-empty subset $A$ of a ternary semigroup $T$ is called a ternary subsemigroup of $T$ if $AAA = A^3 \subseteq A$. By a left (right, lateral) ideal of $T$ we mean a non-empty subset $A$ of $T$ such that $TTA \subseteq A$ ($ATT \subseteq A$, $TAT \subseteq A$) and an ideal is that which is a left, a right and a lateral ideal of $T$. A ternary subsemigroup $A$ of $T$ is called a bi-ideal of $T$ if $ATATA \subseteq A$.

**Definition 2.2.** [8] A bipolar fuzzy subset (briefly, $BF$-subset) $B$ in a non-empty set $X$ is an object having the form

$$B = \{ (x, \mu_B^+(x), \mu_B^-(x)) \mid x \in X \}.$$ 

Where $\mu_B^+ : X \rightarrow [0, 1]$ and $\mu_B^- : X \rightarrow [-1, 0]$.

The positive membership degree $\mu_B^+$ denote the satisfaction degree of an element $x$ to the property corresponding to a $BF$-subset $B$, and the negative membership degree $\mu_B^-$ denotes the satisfaction degree of $x$ to some implicit counter property of $BF$-subset $B$. Bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other, see [9].

**Definition 2.3.** Let $B = (\mu_B^+, \mu_B^-)$ be a bipolar fuzzy set and $(s, t) \in [-1, 0] \times [0, 1]$. Define:
1) the sets $\mathcal{B}_t^+ = \{x \in X \mid \mu^+_B(x) \geq t\}$ and $\mathcal{B}_s^- = \{x \in X \mid \mu^-_B(x) \leq s\}$, which are called positive t-cut of $\mathcal{B} = (\mu^+_B, \mu^-_B)$ and the negative s-cut of $\mathcal{B} = (\mu^+_B, \mu^-_B)$, respectively.

2) the sets $\mathcal{B}_t^+ = \{x \in X \mid \mu^+_B(x) > t\}$ and $\mathcal{B}_s^- = \{x \in X \mid \mu^-_B(x) < s\}$, which are called strong positive t-cut of $\mathcal{B} = (\mu^+_B, \mu^-_B)$ and the strong negative s-cut of $\mathcal{B} = (\mu^+_B, \mu^-_B)$, respectively.

3) the set $X_{\mathcal{B}}^{(s,t)} = \{x \in X \mid \mu^+_B(x) \geq t, \mu^-_B(x) \leq s\}$ is called an $(s,t)$-level subset of $\mathcal{B}$.

4) the set $\mathcal{B}_s^2 = \{x \in X \mid \mu^+_B(x) > t, \mu^-_B(x) < s\}$ is called a strong $(s,t)$-level subset of $\mathcal{B}$.

### 3 Bipolar $(\lambda, \delta)$-Fuzzy Ideals in Ternary Semigroups

In this section, we will define the notion of bipolar $(\lambda, \delta)$-fuzzy ideals in ternary semigroups and discuss some properties of these ideals.

In what follows, let $\lambda, \delta \in [0, 1]$ be such that $0 \leq \lambda < \delta \leq 1$. Both $\lambda$ and $\delta$ are arbitrary but fixed.

**Definition 3.1.** A bipolar fuzzy subset $\mathcal{B} = (\mu^+_B, \mu^-_B)$ of a ternary semigroup $T$ is called a bipolar $(\lambda, \delta)$-fuzzy ternary subsemigroup of $T$ if

1. $\max \{\mu^+_B(xyz), \lambda\} \geq \min \{\mu^+_B(x), \mu^-_B(y), \mu^+_B(z), \delta\}$
2. $\min \{\mu^-_B(xyz), -\lambda\} \leq \max \{\mu^-_B(x), \mu^-_B(y), \mu^-_B(z), -\delta\}$

for all $x, y, z \in T$.

**Definition 3.2.** A bipolar fuzzy subset $\mathcal{B} = (\mu^+_B, \mu^-_B)$ of a ternary semigroup $T$ is called a bipolar $(\lambda, \delta)$-fuzzy left (right, lateral) ideal of $T$ if

1. $\max \{\mu^+_B(xyz), \lambda\} \geq \min \{\mu^+_B(z), \delta\}
   \left(\begin{array}{c}
   \max \{\mu^+_B(xyz), \lambda\} \geq \min \{\mu^+_B(x), \delta\}, \\
   \max \{\mu^+_B(xyz), \lambda\} \geq \min \{\mu^+_B(y), \delta\},
   \end{array}\right)$
2. $\min \{\mu^-_B(xyz), -\lambda\} \leq \max \{\mu^-_B(z), -\delta\}
   \left(\begin{array}{c}
   \min \{\mu^-_B(xyz), -\lambda\} \leq \max \{\mu^-_B(x), -\delta\}, \\
   \min \{\mu^-_B(xyz), -\lambda\} \leq \max \{\mu^-_B(y), -\delta\},
   \end{array}\right)$

for all $x, y, z \in T$.

A bipolar fuzzy subset $\mathcal{B} = (\mu^+_B, \mu^-_B)$ of a ternary semigroup $T$ is called a bipolar $(\lambda, \delta)$-fuzzy ideal of $T$ if it is a bipolar $(\lambda, \delta)$-fuzzy left ideal, bipolar $(\lambda, \delta)$-fuzzy right ideal and bipolar $(\lambda, \delta)$-fuzzy lateral ideal of $T$.

**Example 3.3.** Let $T = \{1, 2, 3, 4\}$ be a ternary semigroup with the follow-
Define a bipolar fuzzy subset \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) in \( T \) as follows:

\[
\mu^+_B(x) = \begin{cases} 
0.6 & \text{if } x = 1 \\
0.4 & \text{if } x = 2, 3, 4
\end{cases}
\quad \text{and} \quad
\mu^-_B(x) = \begin{cases} 
-0.7 & \text{if } x = 1 \\
-0.5 & \text{if } x = 2, 3, 4
\end{cases}
\]

By routine calculations, it can be seen that the bipolar fuzzy set \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) is a bipolar \((0.2, 0.3)\)-fuzzy ideal of \( T \).

**Definition 3.4.** A bipolar fuzzy subset \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) of a ternary semigroup \( T \) is called a bipolar \((\lambda, \delta)\)-fuzzy generalized bi-ideal of \( T \) if

1. \( \max \{ \mu^+_B(xuyvz), \lambda \} \geq \min \{ \mu^+_B(x), \mu^+_B(y), \mu^+_B(z), \delta \} \),
2. \( \min \{ \mu^-_B(xuyvz), -\lambda \} \leq \max \{ \mu^-_B(x), \mu^-_B(y), \mu^-_B(z), -\delta \} \),

for all \( x, y, z, u, v \in T \).

**Definition 3.5.** A bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) of a ternary semigroup \( T \) is called a bipolar \((\lambda, \delta)\)-fuzzy bi-ideal of \( T \) if

1. \( \max \{ \mu^+_B(xuyvz), \lambda \} \geq \min \{ \mu^+_B(x), \mu^+_B(y), \mu^+_B(z), \delta \} \),
2. \( \min \{ \mu^-_B(xuyvz), -\lambda \} \leq \max \{ \mu^-_B(x), \mu^-_B(y), \mu^-_B(z), -\delta \} \),

for all \( x, y, z, u, v \in T \).

**Theorem 3.6.** A bipolar fuzzy subset \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) of a ternary semigroup \( T \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \) if and only if \( \emptyset \neq \mathcal{T}^{(t,s)}_B \) is a ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \) for all \((s, t) \in [-\delta, -\lambda] \times (\lambda, \delta] \).

**Proof.** Let \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) be a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \). Let \( x, y, z \in T \), \((s, t) \in [-\delta, -\lambda] \times (\lambda, \delta] \) and \( x, y, z \in \mathcal{T}^{(t,s)}_B \). Then \( \mu^+_B(x) \geq t \), \( \mu^+_B(y) \geq t \) and \( \mu^+_B(z) \geq t \) also \( \mu^-_B(x) \leq s \), \( \mu^-_B(y) \leq s \) and \( \mu^-_B(z) \leq s \). As \( \mathcal{B} = (\mu^+_B, \mu^-_B) \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \). Therefore,

\[
\max \{ \mu^+_B(xyz), \lambda \} \geq \min \{ \mu^+_B(x), \mu^+_B(y), \mu^+_B(z), \delta \} \geq \min \{ t, t, t, \delta \} = t,
\]

and

\[
\min \{ \mu^-_B(xyz), -\lambda \} \leq \max \{ \mu^-_B(x), \mu^-_B(y), \mu^-_B(z), -\delta \} \leq \max \{ s, s, s, -\delta \} = s.
\]
This implies that $\mu_B^+ (xyz) \geq t$ and $\mu_B^- (xyz) \leq s$. Thus $xyz \in T_B^{(t,s)}$. Hence $T_B^{(t,s)}$ is a ternary subsemigroup of $T$.

Conversely, suppose that $T_B^{(t,s)}$ is a ternary subsemigroup of $T$. Let $x, y, z \in T$ such that

$$\max \{ \mu_B^+ (xyz), \lambda \} < \min \{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \},$$

and

$$\min \{ \mu_B^- (xyz), -\lambda \} > \max \{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \}.$$ 

Then there exist $(s, t) \in [-\delta, -\lambda) \times (\lambda, \delta]$ such that

$$\max \{ \mu_B^+ (xyz), \lambda \} < t \leq \min \{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \},$$

and

$$\min \{ \mu_B^- (xyz), -\lambda \} > s \geq \max \{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \}.$$ 

This shows that $\mu_B^+ (x) \geq t$, $\mu_B^+ (y) \geq t$, $\mu_B^+ (z) \geq t$ and $\mu_B^+ (xyz) < t$, also $\mu_B^-(x) \leq s$, $\mu_B^- (y) \leq s$, $\mu_B^- (z) \leq s$ and $\mu_B^- (xyz) > s$. Thus $x, y, z \in T_B^{(t,s)}$, since $T_B^{(t,s)}$ is a ternary subsemigroup of $T$. Therefore $xyz \in T_B^{(t,s)}$, but this is a contradiction to $\mu_B^+ (xyz) < t$ and $\mu_B^- (xyz) > s$. Thus,

$$\max \{ \mu_B^+ (xyz), \lambda \} \geq \min \{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \},$$

and

$$\min \{ \mu_B^- (xyz), -\lambda \} \leq \max \{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \}.$$ 

Hence $B = (\mu_B^+, \mu_B^-)$ is a bipolar $(\lambda, \delta)$-fuzzy ternary subsemigroup of $T$. The other cases can be seen in a similar way. \hfill $\square$

**Corollary 3.7.** Every bipolar fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal $B = (\mu_B^+, \mu_B^-)$ is a bipolar $(\lambda, \delta)$-fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of $T$ with $\lambda = 0$ and $\delta = 1$.

**Theorem 3.8.** If a bipolar fuzzy subset $B = (\mu_B^+, \mu_B^-)$ is a bipolar $(\lambda, \delta)$-fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of $T$. Then the set $B_\lambda = (\geq B_\lambda, \leq B_\lambda)$ is a ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of $T$, where $\geq B_\lambda = \{ x \in T \mid \mu_B^+ (x) > \lambda \}$ and $\leq B_\lambda = \{ x \in T \mid \mu_B^- (x) < -\lambda \}$.

**Proof.** Suppose that $B = (\mu_B^+, \mu_B^-)$ is a bipolar $(\lambda, \delta)$-fuzzy ternary subsemigroup of $T$. Let $x, y, z \in T$ such that $x, y, z \in B_\lambda$. Then $\mu_B^+ (x) > \lambda$, $\mu_B^- (y) > \lambda$,
\[ \mu_B^+(z) > \lambda \text{ and } \mu_B^-(x) < -\lambda, \mu_B^-(y) < -\lambda, \mu_B^-(z) < -\lambda. \]

Since \( \mathcal{B} = (\mu_B^+, \mu_B^-) \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup therefore,

\[
\max \left\{ \mu_B^+ (xyz), \lambda \right\} \geq \min \left\{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \right\} \\
\geq \min \left\{ \lambda, \lambda, \lambda, \delta \right\} = \lambda,
\]

and

\[
\min \left\{ \mu_B^- (xyz), -\lambda \right\} \leq \max \left\{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \right\} \\
< \max \left\{ -\lambda, -\lambda, -\lambda, -\delta \right\} = -\lambda.
\]

Hence \( \mu_B^+(xyz) > \lambda \) and \( \mu_B^-(xyz) < -\lambda \). This shows that \( xyz \in \mathcal{B}_\lambda \). Hence \( \mathcal{B}_\lambda \) is a ternary subsemigroup of \( T \). The other cases can be seen in a similar way.

\[ \square \]

**Theorem 3.9.** A non-empty subset \( A \) of \( T \) is a ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \) if and only if the bipolar fuzzy subset \( \mathcal{B} = (\mu_B^+, \mu_B^-) \) of \( T \) defined as follows:

\[
\mu_B^+(x) = \begin{cases} 
\geq \delta & \text{if } x \in A \\
\lambda & \text{if } x \notin A,
\end{cases}
\quad \text{and} \quad
\mu_B^-(x) = \begin{cases} 
\leq -\delta & \text{if } x \in A \\
-\lambda & \text{if } x \notin A,
\end{cases}
\]

is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \).

**Proof.** Suppose that \( A \) is a ternary subsemigroup of \( T \). Let \( x, y, z \in T \) be such that \( x, y, z \in A \) then \( xyz \in A \). Hence \( \mu_B^+(xyz) \geq \delta \) and \( \mu_B^-(xyz) \leq -\delta \). Therefore

\[
\max \left\{ \mu_B^+(xyz), \lambda \right\} \geq \delta = \min \left\{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \right\},
\]

and

\[
\min \left\{ \mu_B^- (xyz), -\lambda \right\} \leq -\delta = \max \left\{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \right\}.
\]

If \( x \notin A \) or \( y \notin A \) or \( z \notin A \), then \( \min \left\{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \right\} = \lambda \) and \( \max \left\{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \right\} = -\lambda \). Thus

\[
\max \left\{ \mu_B^+ (xyz), \lambda \right\} \geq \lambda = \min \left\{ \mu_B^+ (x), \mu_B^+ (y), \mu_B^+ (z), \delta \right\},
\]

and

\[
\min \left\{ \mu_B^- (xyz), -\lambda \right\} \leq -\lambda = \max \left\{ \mu_B^- (x), \mu_B^- (y), \mu_B^- (z), -\delta \right\}.
\]

Consequently \( \mathcal{B} = (\mu_B^+, \mu_B^-) \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \).
Conversely: Let \( x, y, z \in A \). Then \( \mu_B^+(x) \geq \delta, \mu_B^+(y) \geq \delta, \mu_B^+(z) \geq \delta \) and \( \mu_B^-(x) \leq -\delta, \mu_B^-(y) \leq -\delta, \mu_B^-(z) \leq -\delta \). As \( B = (\mu_B^+, \mu_B^-) \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \), therefore

\[
\max \{ \mu_B^+(xyz), \lambda \} \geq \min \{ \mu_B^+(x), \mu_B^+(y), \mu_B^+(z), \delta \} \geq \min \{ \delta, \delta, \delta, \delta \} = \delta,
\]

and

\[
\min \{ \mu_B^-(xyz), -\lambda \} \leq \max \{ \mu_B^-(x), \mu_B^-(y), \mu_B^-(z), -\delta \} \leq \max \{ -\delta, -\delta, -\delta, -\delta \} = -\delta.
\]

This implies that \( xyz \in A \). Hence \( A \) is a ternary subsemigroup of \( T \). The other cases can be seen in a similar way.

**Theorem 3.10.** A non-empty subset \( A \) of \( T \) is ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \) if and only if \( B_A = (\mu_{B_A}^+, \mu_{B_A}^-) \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup, left (right, lateral, generalized bi-, bi-) ideal of \( T \).

*Proof.* Let \( A \) be a ternary subsemigroup of \( T \). Then \( B_A = (\mu_{B_A}^+, \mu_{B_A}^-) \) is a bipolar fuzzy ternary subsemigroup of \( T \) and by Corollary 3.7, \( B_A \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \).

Conversely, let \( x, y, z \in T \) be such that \( x, y, z \in A \). Then \( \mu_{B_A}^+(x) = \mu_A^+(y) = \mu_A^+(z) = 1 \) and \( \mu_{B_A}^-(x) = \mu_A^-(y) = \mu_A^-(z) = -1 \). Since \( B_A \) is a bipolar \((\lambda, \delta)\)-fuzzy ternary subsemigroup of \( T \), Therefore

\[
\max \{ \mu_{B_A}^+(xyz), \lambda \} \geq \min \{ \mu_{B_A}^+(x), \mu_{B_A}^+(y), \mu_{B_A}^+(z), \delta \} = \min \{ 1, 1, 1, \delta \} = \delta,
\]

and

\[
\min \{ \mu_{B_A}^-(xyz), -\lambda \} \leq \max \{ \mu_{B_A}^-(x), \mu_{B_A}^-(y), \mu_{B_A}^-(z), -\delta \} = \max \{ -1, -1, -1, -\delta \} = -\delta.
\]

It implies that \( \mu_{B_A}^+(xyz) \geq \delta \) and \( \mu_{B_A}^-(xyz) \leq -\delta \). Thus \( xyz \in A \). Therefore \( B_A = (\mu_{B_A}^+, \mu_{B_A}^-) \) is a ternary subsemigroup of \( T \). The other cases can be seen in a similar way.

**References**


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