Comparison of Differences between Power Means$^{1,2}$

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Abstract: As an application of Mond-Pečarić method, we estimate bounds of operator convexity for convex functions and construct some relations between the power of arithmetic mean (chaotically geometric mean) and the arithmetic mean (chaotically geometric mean) of operator powers, respectively. Besides, we obtain the order relation between arithmetic mean and chaotically geometric mean as follows: If $0 < m \leq A_i \leq M$ with $m < M$ for $i = 1, 2, \cdots, n$, then

$$-L(m, M) \log M h(1) \leq \nabla \alpha(A_1, A_2, \cdots, A_n) - \Diamond \alpha(A_1, A_2, \cdots, A_n) \leq L(m, M) \log M h(1).$$

Keywords: operator convex, arithmetic mean, geometric mean, chaotically geometric mean

1 Preliminary

Throughout this paper, a capital letter means a bounded linear operator on a Hilbert space $H$. An operator $A$ is called positive, in symbol, $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$. $A$ is called strictly positive (simply $A > 0$) if $A$ is positive and invertible. $\alpha_i$ are positive numbers with $\sum_{i=1}^{n} \alpha_i = 1$.

A continuous function $f$ on an interval $I$ is called operator convex on $I$, if $\sigma(A)$,

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\[ \sigma(B) \subset I, \]

then

\[ f((1 - \alpha)A + \alpha B) \leq (1 - \alpha)f(A) + \alpha f(B) \quad \text{holds for } \alpha \in [0, 1]. \quad (1.1) \]

Convex functions and operator convex functions are different. Typical example of such function is \( t^r \) on \((0, \infty)\), which is a convex function for \( r > 2 \), but is not operator convex.

In [1], Ando, Li and Mathias proposed a definition of the geometric mean for an n-tuple of positive operators and showed that it has many required properties on the geometric mean. Following [3,4], we recall the definition of the weighted geometric mean. Following [3,4], we recall the definition of the weighted geometric mean \( G[n, t] \) with \( t \in [0, 1] \) for an n-tuple of positive invertible operators \( A_1, A_2, \cdots, A_n \). Let \( G[2, t](A_1, A_2) = A_1^\frac{1}{t},A_2 \). For \( n \geq 3 \), \( G[n, t] \) is defined inductively as follows: Put \( A_i^{(0)} = A_i \) for all \( i = 1, 2, \cdots, n \), and

\[ A_i^{(r)} = G[n - 1, t](A_j^{(r-1)})_{j \neq i} = G[n - 1, t](A_1^{(r-1)}, \cdots, A_{i-1}^{(r-1)}, A_{i+1}^{(r-1)}, \cdots, A_n^{(r-1)}) \]

inductively for \( r \). Then the sequence \( \{A_i^{(r)}\}_{r=0}^\infty \) have the same limit for all \( i = 1, 2, \cdots, n \) in the Thompson metric. So we can define

\[ G[n, t](A_1, A_2, \cdots, A_n) = \lim_{r \to \infty} A_i^{(r)}. \]

Similarly, we can define the weighted arithmetic mean as follows: Let \( A[2, t](A_1, A_2) = (1 - t)A_1 + tA_2 \) for \( n \geq 3 \), put \( A_i^{(0)} = A_i \) for all \( i = 1, 2, \cdots, n \) and

\[ A_i^{(r)} = A[n - 1, t](A_j^{(r-1)})_{j \neq i} = A[n - 1, t](A_1^{(r-1)}, \cdots, A_{i-1}^{(r-1)}, A_{i+1}^{(r-1)}, \cdots, A_n^{(r-1)}). \]

The sequence \( \{A_i^{(r)}\} \) have the same limit for all \( i = 1, 2, \cdots, n \), so it’s expressed by

\[ A[n, t](A_1, A_2, \cdots, A_n) = \lim_{r \to \infty} A_i^{(r)}. \]

Here we introduce the following means: For positive invertible operators \( A_1, A_2, \cdots, A_n \), define

\[ F(r) = \begin{cases} (\nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r))^{\frac{1}{r}}, & r \neq 0; \\ \exp(\nabla_\alpha(\log A_1, \log A_2, \cdots, \log A_n)), & r = 0. \end{cases} \]

It is clear that \( F(r) \) is monotone increasing under the chaotic order, but is not monotone under the usual order. Besides, \( \nabla_\alpha(A_1, A_2, \cdots, A_n) = F(0) \) is called chaotically geometric mean.

**Theorem A**[6, 7] Let \( 0 < m \leq A_i \leq M \) with \( m < M \) for \( 1, 2, \cdots, n \), and \( \sum_{i=1}^n \|x_i\|^2 = 1 \). If \( f(t) \) is a real valued continuous convex function on \([m, M]\), then

\[ 0 \leq \sum_{i=1}^n (f(A_i)x_i, x_i) - f(\sum_{i=1}^n (A_i x_i, x_i)) \leq \beta(m, M, f), \quad (1.4) \]
where
\[ \beta(m, M, f) = \max \{ \frac{f(M) - f(m)}{M - m}(t - m) + f(m) - f(t); \ t \in [m, M] \}. \] (1.5)

2 Bounds of the operator convexity for convex functions

Based on the previous results coming from the Mond-Pečarić method we obtain the difference type inequalities.

**Theorem 2.1** Let \( 0 < m \leq A_i \leq M \) with \( m < M \) for \( i = 1, 2, \cdots, n \). If \( f(t) \) is a real valued continuous convex function on \([m, M]\), then
\[ -\beta(m, M, f) \leq \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) - \nabla_\alpha(A_1, A_2, \cdots, A_n) \leq \beta(m, M, f). \]

**Proof:** For unit vector \( x \in H \), put \( x_i = \sqrt{\alpha_i}x, i = 1, 2, \cdots, n \) in Theorem A, then we have
\[ \sum_{i=1}^{n} \alpha_i(f(A_i)x, x) \leq f(\sum_{i=1}^{n} \alpha_i(A_ix, x)) + \beta(m, M, f). \]

Since \( f(t) \) is a real valued continuous convex function on \([m, M]\), it follows from (1.4) that
\[ \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n))x, x) = \sum_{i=1}^{n} \alpha_i f(A_i)x, x) \]
\[ \leq f(\sum_{i=1}^{n} \alpha_i A_ix, x) + \beta(m, M, f) \]
\[ \leq (f(\sum_{i=1}^{n} \alpha_i A_ix, x) + \beta(m, M, f). \]

So that
\[ \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)) \leq \beta(m, M, f) + f(\nabla_\alpha(A_1, A_2, \cdots, A_n)). \]

Next, since \( 0 < m \leq \sum_{i=1}^{n} \alpha_i A_i \leq M \) and \( f(t) \) is a real valued continuous convex function on \([m, M]\), we have
\[ \sum_{i=1}^{n} \alpha_i(f(A_i)x, x) \geq \alpha_i \sum_{i=1}^{n} f((A_i)x, x) \geq f(\sum_{i=1}^{n} \alpha_i(A_ix, x)) \]
\[ = f(\nabla_\alpha(A_1, A_2, \cdots, A_n)x, x) \]
\[ \geq f(\nabla_\alpha(A_1, A_2, \cdots, A_n))x, x) - \beta(m, M, f), \]
this gives the required inequality
\[ f(\nabla_\alpha(A_1, A_2, \cdots, A_n)) - \beta(m, M, f) \leq \nabla_\alpha(f(A_1), f(A_2), \cdots, f(A_n)). \]

**Corollary 2.2** Let \(0 < m \leq A_i \leq M\) with \(m < M\). If \(f(t)\) is a real valued continuous concave function on \([m, M]\), then
\[ -\bar{\beta}(m, M, f) \geq \nabla_\alpha(f(A_1), \cdots, f(A_n)) - f(\nabla_\alpha(A_1, \cdots, A_n)) \leq \bar{\beta}(m, m, f), \]
where \(\bar{\beta}(m, M, f) = \min\{\frac{f(M)-f(m)}{M-m}(t-m) + f(m) - f(t); \ t \in [m, M]\} \).

### 3 The main applications

**Theorem 3.1** Let \(0 < m \leq A_i \leq M\) with \(m < M\) for \(i = 1, 2, \cdots, n\).

(i) If \(0 < r \leq 1\), then
\[ \frac{M^r - m^r}{M - m} C(m^r, M^r, \frac{1}{r}) \leq \nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r) - (\nabla_\alpha(A_1, A_2, \cdots, A_n))^r \leq 0; \]

(ii) If \(1 \leq r \leq 2\), then
\[ 0 \leq \nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r) - (\nabla_\alpha(A_1, A_2, \cdots, A_n))^r \leq C(m, M, r); \]

(iii) If \(r > 2\), then
\[ -C(m, M, r) \leq \nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r) - (\nabla_\alpha(A_1, A_2, \cdots, A_n))^r \leq C(m, M, r), \]
where \(C(m, M, r) = \frac{Mm^r-mM^r}{M-m} + (r-1)(\frac{M^r-m^r}{r(M-m)})^\frac{r}{r-1}; \)

**Proof:** Put \(f(t) = t^r\), then, when \(0 < r \leq 1\), \(\bar{\beta}(m, M, f) = \frac{m^r-M^r}{M-m} C(m^r, M^r, \frac{1}{r})\), when \(r > 1\), \(\beta(m, M, f) = C(m, M, r)\). We distinguish three cases as follows:

In the case of \(0 < r \leq 1\), (i) follows from Corollary 2.2 and the definition of operator concave. In the case of \(1 < r \leq 2\), it follows from Theorem 2.1 and operator convex of \(t^r\) that (ii) holds. In the case of \(r > 2\), (iii) stems from Theorem 2.1.

**Theorem 3.2** Let \(0 < m \leq A_i \leq M\) with \(m < M\) for \(i = 1, 2, \cdots, n\).

(i) If \(0 < r \leq 1\leq s\), then
\[ -C(m^r, M^r, \frac{1}{r}) \leq F(s) - F(r) \leq C(m^r, M^r, \frac{1}{r}) + \frac{M - m}{M^s - m^s} C(m, M, s); \]

(ii) If \(0 < 1 \leq r \leq s\), then
\[ -\frac{M - m}{M^r - m^r} C(m, M, r) \leq F(s) - F(r) \leq \frac{M - m}{M^s - m^s} C(m, M, s); \]
(iii) If $0 < r \leq s \leq 1$, then
\[
|F(s) - F(r)| \leq C(m^r, M^r, \frac{1}{r}) + C(m^s, M^s, \frac{1}{s}).
\]

**Proof:** (i) If $0 < r \leq 1$, it follows from (ii) and (iii) of Theorem 3.1 that
\[
-C(m, M, \frac{1}{r}) \leq \nabla_\alpha(A_1^m, A_2^m, \cdots, A_n^m) - \nabla_\alpha(A_1, A_2, \cdots, A_n)^\frac{1}{s} \leq C(m, M, \frac{1}{r}).
\]
Replacing $0 < m \leq A_i \leq M$ by $0 < m^r \leq A_i^r \leq M^r$, we have
\[
-C(m^r, M^r, \frac{1}{r}) \leq \nabla_\alpha(A_1, A_2, \cdots, A_n) - \left(\nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r)\right)^\frac{1}{s} \leq C(m^r, M^r, \frac{1}{r}).
\]

(3.1)

If $s \geq 1$, then $\frac{1}{s} \leq 1$, Theorem 3.1 (i) implies that
\[
-\frac{M^\frac{1}{2} - m^\frac{1}{2}}{M - m}C(m^\frac{1}{2}, M^\frac{1}{2}, s) \leq \nabla_\alpha(A_1^m, A_2^m, \cdots, A_n^m) - \left(\nabla_\alpha(A_1, A_2, \cdots, A_n)^r\right)^\frac{1}{s} \leq 0,
\]
since $0 < m^s \leq A_i^s \leq M^s$, we have
\[
-\frac{M - m}{M^s - m^s}C(m, M, s) \leq \nabla_\alpha(A_1, A_2, \cdots, A_n) - \left(\nabla_\alpha(A_1^s, A_2^s, \cdots, A_n^s)^r\right)^\frac{1}{s} \leq 0. \tag{3.2}
\]

Following from (3.1) and (3.2), we have
\[
-C(m^r, M^r, \frac{1}{r}) \leq \nabla_\alpha(A_1, A_2, \cdots, A_n) - \left(\nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r)^r\right)^\frac{1}{r} \leq 0
\]
as required.

(ii) If $0 < 1 \leq r \leq s$, then $0 < \frac{1}{s} \leq \frac{1}{r} < 1$, it follows from (3.2) that
\[
-\frac{M - m}{M^s - m^s}C(m, M, s) \leq \nabla_\alpha(A_1, A_2, \cdots, A_n) - \left(\nabla_\alpha(A_1^s, A_2^s, \cdots, A_n^s)^r\right)^\frac{1}{r} \leq 0,
\]

\[
-\frac{M - m}{M^r - m^r}C(m, M, r) \leq \nabla_\alpha(A_1, A_2, \cdots, A_n) - \left(\nabla_\alpha(A_1^r, A_2^r, \cdots, A_n^r)^r\right)^\frac{1}{r} \leq 0.
\]
Therefore, we have
\[
- \frac{M-m}{M^r - m^r} C(m, M, r) \leq (\nabla_{\alpha}(A_1^*, A_2^*, \ldots, A_n^*))^{\frac{1}{r}} - (\nabla_{\alpha}(A_1', A_2', \ldots, A_n'))^{\frac{1}{r}}
\]
\[
\leq \nabla_{\alpha}(A_1, A_2, \ldots, A_n) + \frac{M-m}{M^s - m^s} C(m, M, s) - \nabla_{\alpha}(A_1, A_2, \ldots, A_n)
\]
\[
= \frac{M-m}{M^s - m^s} C(m, M, s),
\]

(iii) If \(0 < r \leq s \leq 1\), then \(1 < \frac{1}{s} \leq \frac{1}{r}\), it follows from (3.1) that
\[
-C(m^r, M^r, \frac{1}{r}) \leq \nabla_{\alpha}(A_1, A_2, \ldots, A_n) - (\nabla_{\alpha}(A_1', A_2', \ldots, A_n'))^{\frac{1}{r}} \leq C(m^r, M^r, \frac{1}{r})
\]
\[
-C(m^s, M^s, \frac{1}{s}) \leq \nabla_{\alpha}(A_1, A_2, \ldots, A_n) - (\nabla_{\alpha}(A_1^*, A_2^*, \ldots, A_n^*))^{\frac{1}{s}} \leq C(m^s, M^s, \frac{1}{s}).
\]

Therefore, we have
\[
-C(m^r, M^r, \frac{1}{r}) - C(m^s, M^s, \frac{1}{s}) \leq (\nabla_{\alpha}(A_1^*, A_2^*, \ldots, A_n^*))^{\frac{1}{r}} - (\nabla_{\alpha}(A_1', A_2', \ldots, A_n'))^{\frac{1}{s}}
\]
\[
\leq C(m^r, M^r, \frac{1}{r}) + C(m^s, M^s, \frac{1}{s}).
\]

**Theorem 3.3** If \(0 < m \leq A_i \leq M\) with \(m < M\) for \(i = 1, 2, \ldots, n\), then
\[
-L(e^m, e^M) \log M(e^{M-m}) \leq \nabla_{\alpha}(e^{A_1}, e^{A_2}, \ldots, e^{A_n}) - e^{\nabla_{\alpha}(A_1, A_2, \ldots, A_n)} \leq L(e^m, e^M) \log M(e^{M-m}),
\]
where
\[
L(e^m, e^M) = \frac{e^M - e^m}{M - m}, \quad M(e^{M-m}) = \frac{(e^{M-m} - 1)(e^M - e^m)^{\frac{1}{M-m}-1}}{e \log e^{M-m}}.
\]

**Proof:** Put \(f(t) = e^t\), it follows from Theorem 2.1 that
\[
\beta(m, M, e^t) = \max\{\frac{e^M - e^m}{M - m}(t-m)+e^m-e^t; \ t \in [m, M]\} = L(e^m, e^M) \log M(e^{M-m}),
\]
then
\[
-L(e^m, e^M) \log M(e^{M-m}) \leq \nabla_{\alpha}(e^{A_1}, e^{A_2}, \ldots, e^{A_n}) - e^{\nabla_{\alpha}(A_1, A_2, \ldots, A_n)} \leq L(e^m, e^M) \log M(e^{M-m}).
\]

**Theorem 3.4** If \(0 < m \leq A_i \leq M\) with \(m < M\) for \(i = 1, 2, \ldots, n\), then
\[
-L(m, M) \log M_h(1) \leq \nabla_{\alpha}(A_1, A_2, \ldots, A_n) - \hat{\nabla}_{\alpha}(A_1, A_2, \ldots, A_n) \leq L(m, M) \log M_h(1)
\]
where \(L(m, M) = \frac{M-m}{\log M - \log m}, \ h = \frac{M}{m}.\)
Proof: Replace $A_i$ by $\log A_i$ in theorem 3.3, then $h = \frac{M}{m}$, $L(e^{\log m}, e^{\log M}) = L(m, M)$, as well as the following inequalities hold.

$$-L(m, M) \log M_h(1) \leq \nabla \alpha(e^{\log A_1}, e^{\log A_2}, \ldots, e^{\log A_n}) - \exp(\nabla \alpha(\log A_1, \log A_2, \ldots, \log A_n))$$

$$\leq L(m, M) \log M_h(1).$$

Which shows that

$$-L(m, M) \log M_h(1) \leq \nabla \alpha(A_1, A_2, \ldots, A_n) - \hat{\nabla} \alpha(A_1, A_2, \ldots, A_n) \leq L(m, M) \log M_h(1).$$

References


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