On Bornological Divisors of Zero and Permanently Singular Elements in Multiplicative Convex Bornological Jordan Algebras

Abdelaziz Tajmouati

Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Marhaz
Fez, Morocco
abdelaziztajmouati@yahoo.fr

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Abstract

In this paper we extended naturally the notion of topological divisors of zero (t.d.z) to the case of multiplicative convex bornological algebras. We prove that every element in topological boundary of the set of invertible elements is bornological divisor of Zero. On the other hand we give a characterization of bornological divisor of Zero, some results (Guelfand-Mazur, ...) are given.

Keywords: Multiplicatively convex bornological algebras. Bornological Jordan algebras. Topological divisors of zero. Bornological divisors of Zero. Permanently singular elements

1 Introduction

In [10] (see also [2]) we have already study the topological divisor of zero (t.d.z) in Jordan-Banach algebra. In this work we propose extend this notion to the case of multiplicative convex bornological Jordan algebras. The idea is to use the notion of topological divisor of zero in Jordan-Banach algebra, we have necessary to inject every sequence \((x_n)_{n \geq 0}\) in multiplicative convex bornological complete Jordan algebra to Jordan Banach algebra.
is not true in general, the following example shows this.

Let a increasing sequence of Jordan Banach algebras \((E_n, \| \cdot \|_n)_{n \geq 0}\), we suppose that the sequence \((\| \cdot \|_n)_{n \geq 0}\) is increasing.

Let \(E = \bigcup_{n \geq 0} E_n\), In\(E\) we take the bornology defined by: \(B \subset E\) is bounded in \(E\) if, and only if, there exists \(n \geq 1\) such that \(B\) is bounded in \(E_n\). Then \(E\) is multiplicative convex bornological complete Jordan algebra.

Let the sequence \((x_n)_{n \geq 0}\) such that, every \(n \geq 0\), \(x_n \in E_n\) and \(x_n \notin E_{n-1}\). Then, \((x_n)_{n \geq 0}\) is not in \(E_n\).

On other hand, if we consider a multiplicative convex bornological complete Jordan algebra of type \(M_1\), then for every sequence \((x_n)_{n \geq 0}\) there exists a bounded \(B\) of \(E\) such that \((x_n)_{n \geq 0} \subset E_B\). Indeed, let \(E = \lim_{\rightarrow B} E_B\) multiplicative convex bornological Jordan algebra of type \(M_1\) and let \((x_n)_{n \geq 0} \subset E\). Then, for every \(n \geq 0\), there exists \(B_n \in \mathcal{B}\) such that \(x_n \in E_{B_n}\). Since \(E\) is the type \(M_1\), there exists a sequence \((\lambda_n)_{n \geq 0} \subset \mathbb{R}^+\) such that \(\bigcup_{n \geq 0} \lambda_n B_n\) is a bounded in \(E\). Then there is \(\alpha \in \mathbb{R}^+\) and \(B \in \mathcal{B}\) such that \(\bigcup_{n \geq 0} \lambda_n B_n \subset \alpha B\). Consequently, for every \(n \geq 0\), \(E_{B_n} \subset E \bigcup_{n \geq 0} \lambda_n B_n \subset E_B\). This gives, \((x_n)_{n \geq 0} \subset E_B\).

Observe that every Jordan Banach algebra is a multiplicative convex bornological complete Jordan algebra of type \(M_1\). Also, if \(E\) is unital topological Jordan algebra with continuous inverse such that it’s F-space, then \(E\) is a multiplicative convex bornological complete Jordan algebra of type \(M_1\).

2 Preliminaries

Throughout this work, we suppose that the field \(\mathbb{K}\) is either the real \(\mathbb{R}\) or the complex field \(\mathbb{C}\).
Recall that a bornology on a set \(X\) is a family \(\mathcal{B}\) of subset \(X\) such that \(\mathcal{B}\) is a covering of \(X\), hereditary under inclusion and stable under finite union.
The pair \((X, \mathcal{B})\) is called bornological set.
A subfamily \(\mathcal{B}'\) of \(\mathcal{B}\) is said to be base of bornology \(\mathcal{B}\), if every element of \(\mathcal{B}\) is contained in an element of \(\mathcal{B}'\).
A bornology \(\mathcal{B}\) on \(\mathbb{K}\)-vector space \(E\) is said to be vector bornology on \(E\) if the maps \((x, y) \mapsto x + y\) and \((\lambda, x) \mapsto \lambda x\) are bounded.
We called a bornological vector space(b.v.s) any pair \((X, \mathcal{B})\) consisting of a vector space \(E\) and a vector bornology \(\mathcal{B}\) on \(E\).
A (b.v.s) space is called of type $M_1$ if, its satisfy the countability condition of Mackey: *For every sequence of bounded $(B_k)_k$ in $E$, there exists a sequence of scalars $(\lambda_k)_{k \geq 0}$ such that $\bigcup_{k=0}^{\infty} \lambda_k B_k$ is bounded in $E$.*

A vector bornology on a vector space is called a convex vector bornology if it is stable under the formation of convex hull.

A bornological vector space is said a convex bornological vector space (cbvs) if its bornology is convex.

A sequence $(x_n)_{n \geq 0}$ in bornological vector space (b.v.s) $E$ is said Macky-convergent to 0 (or converge bornological to 0) if there exists a bounded set $B \subset E$ such that

$$\forall \epsilon > 0 \ \exists n_0 \in \mathbb{N}, n \geq n_0 \implies x_n \in \epsilon B.$$ 

If $E$ is (cbvs), then $(x_n)_{n \geq 0}$ is Macky-convergent to 0 if there exists a bounded disk $B \subset E$ such that $(x_n)_{n \geq 0} \subset E_B$ and $(x_n)_{n \geq 0}$ converges to 0 in $E_B$, where $(E_B, p_B)$ is the vector space spanned by $B$ and endowed with the semi-norm $p_B$ gauge of $B$.

A set $B$ in a (bvs) $E$ is said $M$-closed (or b-closed) if every sequence $(x_n)_{n \geq 0} \subseteq B$ Macky convergent in $E$ its limit belongs in $B$.

Let $E$ be a (cbvs) and $A$ is a disk in $E$. $A$ is called a completant disk if the space $(E_A, p_A)$ spanned by $A$ and semi-normed by the gauge of $A$ is a Banach space.

A (cbvs) $E$ is called a complete convex bornological vector space if its bornology has a base consisting of completant disks. For detail see [5], [6] and [7].

An algebra over $\mathbb{K}$ is a $\mathbb{K}$-vector space $A$ with a bilinear map $(x, y) \mapsto x.y$ of $A \times A$ into $A$. If this product is associative (resp commutative), we say that the algebra is associative (resp. commutative).

Let $x \in A$, define the following map

$$U_x(y) = 2x(xy) - x^2y, \forall y \in A$$

It is well-known that $x \mapsto U_x$ is quadratic.

An algebra $A$ is a Jordan algebra if every $(x, y) \in A^2$ we have $xy = yx$ and $(x^2y)x = x^2y$.

Let $A$ be an algebra, we denote by $A^+$ the algebra $A$ equipped with its vector space structure and the product $\circ$ defined by : $x \circ y = \frac{1}{2}(xy + yx)$ for all $x, y \in A$. The product $\circ$ is called the Jordan product.

$(A^+, +, \cdot, \circ)$ is Jordan algebra.

An normed Jordan algebra $A$ is a Jordan algebra equipped by a norm $\| \|$ satisfying $\|xy\| \leq \|x\|\|y\|$ for every $(x, y) \in A^2$, if $(A, \| \|)$ is Banach space then $A$ is called a Jordan Banach algebra.
Let $A$ be a Jordan Banach algebra with unit $e$, if $x \in A$ such that $\|x - e\| \leq 1$ then $x$ is invertible.

Let $A$ be an algebra, Recall that a vector bornology $\mathcal{B}$ on $A$ is called algebra bornology if the usual product is bounded in $A \times A$, in this case $(A, \mathcal{B})$ is called bornological Jordan algebra. If again $\mathcal{B}$ have a pseudo basis $\mathcal{B}'$ of idempotent disc, $(A, \mathcal{B})$ is called convex multiplicative bornological Jordan algebra (cmbJa).

It is well known if $(A, \mathcal{B})$ is (cmbJa) then $(A, \mathcal{B})$ is bornological inductive limit of semi normed Jordan algebras $(A_B, \mathcal{B}_B)_{B \in \mathcal{B}}$.

Recall that a (cmbJa) $A$ is separated (respectively complete) if, and only if it’s bornological inductive limit of normed Jordan algebras (respectively bornological inductive limit of Jordan Banach algebras).

Let $(A, \mathcal{B})$ a unital (cmbJa) and $x \in A$, then $x$ is invertible if, and only if, $x$ is invertible in one $A_B$ which $B \in \mathcal{B}$. If again $(A, \mathcal{B})$ is complete then the group of invertible elements is Mackey open.

Let $A$ a unital Jordan algebra with unit $e$, then the Jacobson radical $\text{Rad}(A)$ of $A$ is defined by (For detail, see [4], [8], and [9])

$$\text{Rad}(A) = \{ x \in A : \forall y \in A, e - U_x(y) \text{ is invertible} \}$$

### 3 Definitions and Properties

**Definition 3.1.** Let $(E, \mathcal{B})$ a complete (cmbJa) and $x \in E$.

$x$ is said a bornological divisor of zero (bdz), if there exists $B \in \mathcal{B}$ such that $x$ is a topological divisor of zero (tdz) in Jordan Banach algebra $(E_B, P_B)$.

**Definition 3.2.** Let $E$ be a Jordan Banach algebra and $x \in E$.

$x$ is called an absolute divisor of zero (a.d.z) if there is an element $y \neq 0$ of $E$ such that $U_x(y) = 0$.

For example every (a.d.z) in complete (cmbJa) is a (bdz).

**Remark 1.** In Jordan Banach algebra, the notion of (tdz) and (bdz) coincide.

**Proposition 3.1.** Let $(E, \mathcal{B})$ a complete (cmbJa) and $z$ some (bdz).

Then, we have:

1) For every $n \geq 0$, $z^n$ is a (bdz)

2) For every $a \in E$, $U_z(a)$ is a (bdz)
Proof: Applique [10, proposition 5.1, and proposition 5.2]

**Proposition 3.2.** Let \((E, \mathcal{B})\) an unital complete \((cmbJa)\) of type \(M_1\). Then. Every non invertible element which is limit on Mackey sense of a sequence of invertible elements in \(E\) is a \((bdz)\).

On other expression, \(Fr(G(E)) \subset Z_b\), where \(G(E)\) is a groupe of invertible elements of \(E\), \(Fr\) is the topological boundary of M-closeness and \(Z_b\) the set of a \((bdz)\) of \(E\).

For proof, we need the necessary following lemma.

**Lemma 3.1.** Let \((E, \mathcal{B})\) a unital complete \((cmbJa)\) of type \(M_1\). Let \((x_n)_{n \geq 0}\) a sequence of invertible elements in \(E\) which converges of Mackey-sene to \(x\). If \((x_n^{-2})_{n \geq 0}\) is bounded, then \(x\) is invertible element in \(E\).

Proof: Let \(e\) the uint of \(E\). we have

\[
(\forall n \geq 0), U_{x_n}(x_n^{-2}) = e
\]

Then

\[
U_x(x_n^{-2}) - e = (U_x - U_{x_n})(x_n^{-2})
\]

Since \((x_n^{-2})_{n \geq 0}\) is bounded, there exists \(\alpha > 0\) and \(B_1 \in \mathcal{B}\) such that \((x_n^{-2})_{n \geq 0} \subset \alpha B_1\).

On other hand \((x_n)_{n \geq 0}\) converges of Mackey sense to an element \(x\) in \(E\), then there exists \(B_2 \in \mathcal{B}\) such that \((x_n)_{n \geq 0} \subset E_{B_2}, x \in E_{B_2}\) and

\[
\lim_{n \to \infty} p_{B_2}(x_n - x) = 0.
\]

\(\mathcal{B}\) is growth filtrating, then there exists \(B \in \mathcal{B}\) such that: \((x_n^{-2})_{n \geq 0} \subset E_B, (x_n)_{n \geq 0} \subset E_B, x \in E_B\) and \(\lim_{n \to \infty} p_B(x_n - x) = 0\).

Since \(z \mapsto U_z\) is continuous in \(E_B\), then \(\lim_{n \to \infty} \|U_{x_n} - U_x\| = 0\). Consequently we have:

\[
\forall n \geq 0 \text{ } \exists N_n \in \mathbb{N}, \text{ } n \geq N_n \Rightarrow \|U_{x_n} - U_x\| \leq 1.
\]

\[
\forall n \geq 0 \text{ } \exists N_n \in \mathbb{N}, \text{ } n \geq N_n \text{ and } p_B(t) \leq 1 \Rightarrow p_B(U_{x_n}(t) - U_x(t)) \leq 1.
\]

Since for every \(n\), \(p_B(\frac{1}{\alpha}x_n^{-2}) \leq 1\) then for \(n\) sufficiently large
\[ p_B(U_{x_n} \left( \frac{1}{\alpha} x_n^{-2} \right) - U_x \left( \frac{1}{\alpha} x_n^{-2} \right) ) \leq 1 \quad \text{implies} \quad p_B(U_x \left( \frac{1}{\alpha} x_n^{-2} \right) - e) \leq 1. \]

Then, for \( n \) sufficiently large, \( U_x \left( \frac{1}{\alpha} x_n^{-2} \right) \) is invertible in \( E_B \).

Consequently, \( x \) is invertible in \( E \). \( \blacksquare \)

**Proof of proposition**

Let \( y_n \) the inverse of \( x_n \) for all \( n \). \( (y_n^2) \) is not bounded in \( E \) (Lemma 3.1). Then, it’s not bounded in every \( (E_B, p_B) \). \( E \) being of type \( M_1 \), then there exists \( B_1 \in B \) such that \( (y_n^2) \in E_{B_1} \).

On other hand, \( (x_n)_{n \geq 0} \) bornological converges to \( x \), then there is \( B_2 \in B \) such that \( (x_n)_{n \geq 0} \) and \( x \) belongs in \( E_{B_2} \) and \( \lim_{n \to \infty} p_{B_2}(x_n - x) = 0 \).

\( B \) being growth filtrating, then there exists \( B \in B \) such that \( (y_n^2) \in E_B \), \( (x_n)_{n \geq 0} \subset E_B \), \( x \in E \) and \( \lim_{n \to \infty} p_B(x_n - x) = 0 \).

We can suppose that, for every \( n \geq 0 \), \( y_n^2 \neq 0 \) and \( \lim_{n \to \infty} p_B(y_n^2) = 0 \).

Put \( z_n = \frac{y_n^2}{p_B(y_n^2)} \). We have \( \| z_n \| = 1 \), for all \( n \geq 0 \) and

\[ U_z(z_n) = \frac{y_n^2}{p_B(y_n^2)} + (U_z - U_{x_n})(z_n). \]

Since \( p_B([U_z - U_{x_n}](z_n)] \leq \| U_z - U_{x_n} \| \), we obtain \( \lim_{n \to \infty} U_z(z_n) = 0 \). Then, \( z \) is a (tdz) in \( (E_B, p_B) \).

Consequently, \( z \) is a (bdz) in \( E \). If \([G(E)]^{(1)}\) is the closure of \( G(E) \) for the M-closeness topological of \( E \) we have

\[ [G(E)]^{(1)} \cap [G(E)]^c \subset Z_b. \]

Since \( G(E) \) is M-open. Then \( Fr(G(E)) \subset Z_b \). \( \blacksquare \)

From the above proposition we immediately obtain the followings corollary, which generalize some results in [8].

**Corollary 3.1.** Let \( (E, B) \) an unital complete (cmbJa) of type \( M_1 \).

Let \( x \in E \), then for every scalar \( \lambda \) in bornological boundary of \( sp_E(x) \) the element \( x - \lambda e \) is a (bdz).

**Proof:** Since \( sp_E(x) \) is non-empty closed in \( \mathbb{C} \), therfore

\[ Fr(sp_E(x)) = sp_E(x) \cap ([sp_E(x)]^c) \]

Let \( \lambda \in Fr(sp_E(x)) \), then \( x - \lambda e \) is non-invertible in \( E \) and there exists \( (\lambda_n)_n \subset \mathbb{C} \) converging to \( \lambda \) such that \( x - \lambda_n e \) is invertible in \( E \) for each \( n \geq 0 \).
Hence \((x - \lambda_n e)_n\) converges in Mackey sense to \(x - \lambda e\). Consequently, \(x\) is a (bdz).

**Corollary 3.2.** Let \((E, \mathcal{B})\) an unital complete (cmbJa) of type \(M_1\). Every element in the Jacobson radical of \(E\) is a (bdz).

**Proof:** Let \(x \in \text{Rad}(E)\), then \(sp_E(x) = \{0\}\). Hence \(Fr(sp_E(x)) = \{0\}\). So that \(x\) a (bdz).

**Corollary 3.3.** Let \((E, \mathcal{B})\) an unital complete (cmbJa) of type \(M_1\). Let \(F\) be a closed subalgebra of \(E\) such that \(F\) contains the unit of \(E\), then for every \(x \in F\) we have:

\[
sp_E(x) \subset sp_F(x) \quad \text{and} \quad Fr(sp_F(x)) \subset Fr(sp_E(x))
\]

**Corollary 3.4.** Let \((E, \mathcal{B})\) an unital complete (cmbJa) of type \(M_1\). If \(E\) has not a non-zero (bdz), then \(E\) is bornological isomorphic to \(\mathbb{C}\).

**Proof:** Let \(x \in E \setminus \{0\}\) and \(\lambda \in Fr(sp_E(x))\). Then \(x - \lambda e\) is a (bdz), hence \(x = \lambda e\), this proves the corollary.

The characterization of (bdz) need the following definition.

**Definition 3.3.** Let \((E, \mathcal{B})\) a separated (cmbJa).

For every \(B \in \mathcal{B}\) and \(x \in E_B\), denotes by

\[
\lambda_B(x) = \inf_{\substack{z \in E_B \\
p_B(z) = 1}} p_B(U_x(z))
\]

The following proposition characterize the (bdz).

**Proposition 3.3.** Let \((E, \mathcal{B})\) a complete (cmbJa) and \(x \in E\). \(x\) is a (bdz) if, and only if, there exists \(B \in \mathcal{B}\) such that \(\lambda_B(x) = 0\).

**Proof:** It suffice to applique [10, definition 3.1] and the definition of (bdz) and the characterization of the infimum.

**Proposition 3.4.** Let \((E, \mathcal{B})\) an unital complete (cmbJa) of type \(M_1\). Suppose that there exists a constant \(K \in \mathbb{R}^+\) such that

\[
\forall B \in \mathcal{B} \quad p_B(U_x(y)) \geq K p_B(x)p_B(y) \quad \forall x, y \in E_B.
\]

Then, \(E\) is bornological isomorphic to \(\mathbb{C}\).
Proof:
It suffices to show that $E$ haven't any non-zero (bdz).
Suppose that there is $z \in E$ such that is a (bdz), then there exists $B \in \mathcal{B}$ for which $z$ is a (dtz) in $(E_B, p_B)$. Hence $\lambda_B(z) = 0$.

Since for every $y \in E$ we have $\lambda_B(z) \geq Kp_B(z)$. It follows $p_B(z) = 0$, consequently $z = 0$, and therefore $E$ haven't any non-zero (bdz). This proves that $E$ is bornological isomorphic to $\mathbb{C}$. ■

Definition 3.4. Let $E$ a separated (cmbJa).
An element $x \in E$ is said to satisfy the condition (D.F) if there exists a non-bounded sequence $(x_n)_{n \geq 0}$ in $E$ such that $(U_x(x_n))_{n \geq 0}$ converges in the Mackey-sense to 0.

Proposition 3.5. Let $(E, \mathcal{B})$ an unital complete (cmbJa) of type $M_1$. Then
Every element $x \in E$ satisfying the condition (D.F) is a (bdz).

Proof: Suppose that $x \in E$ satisfies the condition (D.F), then there exists $B_0 \in \mathcal{B}$ such that $x \in E_{B_0}$ and a non-bounded sequence $(x_n)_{n \geq 0} \subset E$ where $(U_x(x_n))_{n \geq 0}$ converges in the Mackey-sense to 0. Therefore, there exists $B_1 \in \mathcal{B}$ such that $(U_x(x_n))_{n \geq 0} \subset E_{B_1}$ converges to 0 in $(E_{B_1}, p_{B_1})$. Since $E$ is of type $M_1$, then there exists $B_2 \in \mathcal{B}$ such that $(x_n)_{n \geq 0} \subset E_{B_2}$. $\mathcal{B}$ being growth filtrating, then there exists $B \in \mathcal{B}$ and $x \in E_B$, $(x_n)_{n \geq 0} \subset E_B$, $(U_x(x_n))_{n \geq 0} \subset E_B$, $(U_x(x_{n+1}))_n$ converges to 0 in $(E_B, p_B)$ and $(x_n)_{n \geq 0}$ is'nt bounded in $(E_B, p_B)$. Therefore, $(x_n)_{n \geq 0}$ not converging to 0 in $(E_B, p_B)$. From which we conclude that $x$ is a (bdz) in $E$. ■

A natural question is given, what can we say for the conversely of the proposition 3.5?
Unfortunately, in general situation, if $x$ is a (bdz) in unital complete (cmbJa) of type $M_1$, then $x$ can not satisfies the condition (D.F) as the following example shows.

Let $A$ the set of all real or complex sequences.
Let $x = (x_n)_{n \geq 0}$ and $y = (y_n)_{n \geq 0}$ two elements in $A$ and $\lambda \in \mathbb{C}$. In $A$ we defined the following usual operations:

\[ x + y = (x_n + y_n)_n, \lambda x = (\lambda x_n)_n \text{ and } x \ast y = (z_n)_n \text{ where } z_n = \sum_{p=0}^{n} x_py_{n-p}. \]

With this operations, $A$ is an unital commutative algebra with unit $e = (e_n)_{n \geq 0}$ such that $e_0 = 1$ and $e_n = 0$ $\forall n \geq 1$.

For all $n \geq 0$, put $p_n(x) = \sum_{i=0}^{n} |x_i|$ where $x = (x_i)_{i \geq 0}$.
$p_n$ is a sub-multiplicative semi-norm on $A$. 

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Indeed, let \( x = (x_n)_{n \geq 0} \) and \( y = (y_n)_{n \geq 0} \) in \( A \). \( x \ast y = (z_n)_{n} \) such that \( z_n = \sum_{p=0}^{n} x_p y_{n-p} \).

Let \( n \geq 0 \), \( p_n(x \ast y) = \sum_{i=0}^{n} |z_i| = |z_0| + |z_1| + \ldots + |z_n| \)
\[
= |x_0 y_0| + |x_0 y_1 + x_1 y_0| + \ldots + |x_0 y_n + x_1 y_{n-1} + \ldots + x_n y_0| \\
\leq (\sum_{i=0}^{n} |x_i|)(\sum_{i=0}^{n} |y_i|) = p_n(x)p_n(y).
\]

With the sequence of semi-norms \( (p_n)_{n \geq 0} \), \( A \) is an unital commutative metrizable complete (cmbJa).

On other hand, the group of invertible elements of \( A \) is exactly
\( \{ x = (x_n)_{n \geq 0} \in A : x_0 \neq 0 \} \), which is open in \( A \). Then, \( A \) is an unital commutative Frechet (cmba) which is Q-algebra. Hence, every element of \( A \) is regular, consequently, \( A \) is unital commutative (cmbJa) complete for Von-Neumann bornology [1].

Since \( A \) is metrizable, then it’s of the type \( M_1 \) [7, p. 226].

Now, Consider \( z = (z_n)_{n \geq 1} \) such that \( z_1 = 1 \) and \( z_n = 0, \forall n \neq 1 \).

We show that \( z \) is a (bdz). Of course, it suffices to prove that \( z \) belongs in the Jacobson radical of \( A \). Let \( y = (y_n)_{n \geq 0} \in A \), the term of row zero in sequence \( e + z \ast y \) is 1. Hence \( e + z \ast y \) is invertible in \( A \). Therefore \( z \) belongs in radical of \( A \). It then follows that \( z \) is a (bdz).

Next, we prove that for every non-bounded \( (x^m)_{m \geq 0} \subset A \), the sequence \( (z \ast x^m)_{m \geq 0} \) not converges to 0 in Mackey-sense.

For all \( m \geq 0 \), set \( x^m = (x^m_n)_{n \geq 0} \).
\[
z \ast x^m = (z_n)_{n \geq 0} + (x^m_n)_{n \geq 0} = (t^m_n), \text{ with } t^0_n = 0 \text{ and } t^m_n = x^m_n, \forall n \geq 0.
\]
Then, the sequence \( (z \ast x^m)_{m \geq 0} \) is not bounded. Consequently, \( (z \ast x^m)_{m \geq 0} \) not converges to 0 in Mackey -sense.

Observe that for Jordan Banach algebra we have the following characterization.

**Proposition 3.6.** Let \( E \) be an Jordan Banach algebra. Let \( x \) be a given element in \( E \). Then
\( x \) is a (tdz) if, and only if, \( x \) satisfies the condition \((D.F)\).

**Proof:** Let \( x \in E \), and suppose that there exists a non-bounded sequence \( (x_n)_{n \geq 0} \) in \( E \) such that \( (U_x(x_n))_{n \geq 0} \) converges bornological to 0. Then, \( (U_x(x_n))_{n \geq 0} \) converges topologically to 0 and \( (x_n)_{n \geq 0} \) not converges to 0 in \( E \). Therefore, \( x \) is a (tdz) in \( E \).

Conversely, if \( x \) is a (tdz) in \( E \), there exists a sequence \( (x_n)_{n \geq 0} \) in \( E \) such that
\[(U_x(x_n))_{n \geq 0} \text{ converges to } 0 \text{ and } (x_n)_{n \geq 0} \text{ not converges to } 0.\]

The map \((x, y) \mapsto \|x - y\|\) is a distance invariant by translation and defines the topology of \(E\). Hence, there exists a sequence \((\gamma_n)_{n \geq 0} \subset \mathbb{R}^+\) such that \(\lim \gamma_n = +\infty\) and \(\lim \gamma_n U_x(x_n) = 0\).

For all \(n \geq 0\) put \(y_n = \gamma_n x_n\). Then, \((y_n)_{n \geq 0}\) is not bounded and \((U_x(y_n))_{n \geq 0}\) converges bornologically to 0 [5].

**Proposition 3.7.** Let \(E = \lim_{\rightarrow B} E_B\) be a complete (cmbJa). Then, every absolute divisor of zero (a.d.z) satisfies the condition (D.F).

**Proof:** Let \(x\) a (a.d.z). Then there exists \(y \neq 0\) such that \(U_x(y) = 0\).

For every \(n \geq 0\), put \(y_n = ny\). Then, \((y_n)_{n \geq 0}\) is not bounded in \(E\) and \(U_x(y_n) = 0\), for all \(n \geq 0\). Therefore, \((U_x(y_n))_{n \geq 0}\) converges bornological to 0.

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4 Permanently singular elements in complete multiplicative convex bornological Jordan algebras

**Definition 4.1.** Let \(E\) an unital separated (cmbJa).

An unital separated (cmbJa) \(E_1\) is called an extension of \(E\), if \(E\) is subalgebra of \(E_1\), the unit element of \(E_1\) is a unit element for \(E\) and the induced bornology of \(E_1\) on \(E\) coincide with the bornology of \(E\).

**Definition 4.2.** Let \(E\) an unital separated (cmbJa).

An element in \(E\) is called permanently singular if it’s not invertible in every extension of \(E\).

**Proposition 4.1.** Let \(E\) an unital complete (cmbJa).

Every element in \(E\) satisfying the condition (D.F) is not invertible in \(E\).

**Proof:** Let \(x \in E\) where \(E = \lim_{\rightarrow B} E_B\) satisfying the condition (D.F). Then, there exists a non-bounded sequence \((x_n)_{n \geq 0} \subset E\) such that \((U_x(x_n))_{n \geq 0}\) converges bornological to 0. Then, there is \(B_0 \in \mathcal{B}\) such that \((U_x(x_n))_{n \geq 0}\) converges to 0 in \((E_{B_0}, p_{B_0})\). Since \(\mathcal{B}\) is growth filtrating, we can suppose that \((x_n)_{n \geq 0} \subset E_{B_0}\).

Suppose that \(x\) is invertible in \(E\). Then, there is \(B_1 \in \mathcal{B}\) such that \(x\) is invertible in \(E_{B_1}\). Then, there is \(B \in \mathcal{B}\) where \(E_{B_0} \subset E_B, E_{B_1} \subset E_B, p_B \leq p_{B_1}, (x_n)_{n \geq 0} \subset E_B, (U_x(x_n))_{n \geq 0} \subset E_B\) and \(\lim_{n \to \infty} p_B(U_x(x_n)) = 0\). On other hand \(x\) is invertible in \(E_B\), hence \(U_x\) is invertible in \(\mathcal{L}(E_B)\) and we have \(U_{x^{-1}} = (U_x)^{-1}\), the map \(z \mapsto U_{x^{-1}}(z)\) is continuous and
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\[ p_B(x_n) = p_B[U_{x^{-1}}(U_x(x_n))], \text{ then } \lim_{n \to \infty} p_B(x_n) = 0. \] We get a contradiction. Hence, \( x \) is not invertible in \( E \). The proof is complete. \( \blacksquare \)

**Proposition 4.2.** Let \( E \) an unital complete (cmbJa).

Let \( x \in E \), if \( x \) satisfies the condition \((D.F)\) in \( E \), then \( x \) is permanently singular in \( E \).

**Proof:** We can prove that if \( x \) satisfies the condition \((D.F)\) in \( E \), then it satisfies this condition in every extension of \( E \). Consequently, \( x \) is permanently singular in \( E \). \( \blacksquare \)

**Proposition 4.3.** Let \( E \) an unital complete (cmbJa).

Let \( x \in E \), if \( x \) satisfies the condition \((D.F)\) in \( E \) and if \( y \in E \), then again the element \( U_x(y) \) satisfies the condition \((D.F)\).

**Proof:** For two elements \( x, y \) in \( E \), we have \( U_{U_x(y)} = U_x U_y U_x \). If \( x \) satisfies the condition \((D.F)\) in \( E \), then there exists a non-bounded sequence \( (x_n)_{n \geq 0} \subset E \) such that \((U_x(x_n))_{n \geq 0}\) converges bornological to 0. Hence there is \( B \in B \) such that \((U_x(x_n))_{n \geq 0}\) converges to 0 in \((E_B, p_B)\). Hence the map \( z \mapsto U_x \circ U_y(z) \) is continuous in \( E_B \), the sequence \((U_x \circ U_y \circ (U_x(x_n)))_{n \geq 0}\) converges to 0 in \( E_B \). Therefore, \((U_{U_x(y)}(x_n))_{n \geq 0}\) converges bornological to 0. We conclude that \( U_x(y) \) satisfies the condition \((D.F)\). \( \blacksquare \)

**References**


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