New Method for Computing the Local Behavior of $L^q$-Bifurcation Curve for Logistic Equations

Tetsutaro Shibata

Laboratory of Mathematics, Institute of Engineering
Hiroshima University, Higashi-Hiroshima, 739-8527, Japan
shibata@amath.hiroshima-u.ac.jp

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Abstract

We consider the logistic equation

$$-u''(t) + f(u(t)) = \lambda u(t), \quad u(t) > 0, \quad t \in I := (0, 1), \quad u(0) = u(1) = 0,$$

where $\lambda > 0$ is a parameter and $f(u) = f_0(u) + g(u)$. We give the simple scheme how to obtain the precise local behavior of $L^q$-bifurcation curve $\lambda_q(\alpha)$. Precisely, if $g(u)$ is regarded as a small perturbation of $f_0(u)$, then we establish the simple and practical scheme how to obtain the asymptotic expansion formula for $\lambda_q(\alpha)$ as $\alpha \to 0$ without using the complicated asymptotic expansion of the solution $u_\alpha$ associated with $\lambda_q(\alpha)$. This scheme is also expected to be useful to compute numerically the local structure of the $L^q$-bifurcation curve.

Mathematics Subject Classification: 34C23, 34B15

Keywords: $L^q$-bifurcation curve, local asymptotic behavior, logistic equation
1 Introduction

We consider the following nonlinear bifurcation problem arising in population dynamics:

\[-u''(t) + f(u(t)) = \lambda u(t), \quad t \in I := (0, 1), \quad (1.1)\]
\[u(t) > 0, \quad t \in I, \quad (1.2)\]
\[u(0) = u(1) = 0, \quad (1.3)\]

where \(\lambda > 0\) is a positive parameter. We assume the following condition (A.1).

(A.1) \(f(u)\) is a \(C^1\)-function for \(0 \leq u \ll 1\), \(f(u) > 0\) for \(u > 0\), and \(f(0) = f'(0) = 0\).

If \(f\) satisfies (A.1), then for any given \(0 < \alpha \ll 1\), there exists a unique solution \((\lambda, u) = (\lambda_q(\alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\bar{I})\) of (1.1)–(1.3) with \(\|u_\alpha\|_q = \alpha\), where \(\|\cdot\|_q\) denotes the \(L^q\)-norm (cf. [1]). The set \(\{(\lambda_q(\alpha), u_\alpha) ; 0 < \alpha \ll 1\}\) gives all solutions of (1.1)–(1.3) near the bifurcation point \((\pi^2, 0)\). We call \(\lambda_q(\alpha)\) the \(L^q\)-local bifurcation curve of positive solutions.

The equation (1.1)–(1.3) and the related problems have been studied intensively by local bifurcation theory of \(L^\infty\)-framework and \(L^2\)-framework, topological methods, and so on (cf. [1–11]). It should be mentioned that (1.1)–(1.3) is motivated by the equation of population density for some species when \(f(u) = u^2\) (cf. [6]). \(\lambda > 0\) is, in this case, regarded as the reciprocal number of its diffusion rate. From this biological viewpoint, it seems important to consider the asymptotic behavior of \(\lambda_1(\alpha)\), which represents the relationship between the total number of population and the reciprocal number of its diffusion rate.

Certainly, by using the classical theory of [9], it is possible to obtain, for example, up to the forth terms of \(\lambda_q(\alpha)\) as \(\alpha \to 0\) theoretically. However, the method in [9] does not seem to be practical to obtain the asymptotic formula for \(\lambda_q(\alpha)\) as \(\alpha \to 0\) easily, since it is essential to obtain the asymptotic expansion of \(u_\alpha\) \((0 < \alpha \ll 1)\) up to the forth term before obtaining the asymptotic expansion formula for \(\lambda_q(\alpha)\) up to the forth term. By this reason, it is rather difficult to obtain the asymptotic formula for \(\lambda_q(\alpha)\) as \(\alpha \to 0\) up to the higher term simply and practically.

Motivated by this, we overcome this difficulty and establish the simple and easy scheme how to calculate the local behavior of bifurcation curve \(\lambda_q(\alpha)\) precisely. This scheme is quite practical to obtain the asymptotic expansion formula for \(\lambda_q(\alpha)\) as \(\alpha \to 0\), since our method does not depend on the asymptotic expansion of \(u_\alpha\) at all. All we need is the elementary and direct calculation of some definite integrals.

Therefore, the further direction of this study is to apply our method to the computation of the asymptotic formulas for bifurcation curves numerically.
Before stating our results, we explain some notations. In what follows, we fix $1 \leq q < \infty$. Let $f(u) := f_0(u) + g(u)$, where $f(u)$ and $f_0(u)$ satisfy (A.1). For convenience, we write $\lambda(\alpha)$ and $\lambda_0(\alpha)$ as the bifurcation curves associated with $f(u)$ and $f_0(u)$, respectively for $0 \leq \alpha \ll 1$.

We assume that $g(u)$ satisfies the following condition (B.1).

(B.1) $g(u) \geq 0$ for $0 \leq u \ll 1$. Furthermore, $g(u)/f_0(u) \to 0$ as $u \to 0$.

Now we state our results.

**Theorem 1.1.** Let $f_0(u) = u^p$ ($p > 1$). Assume that $g(u) = u^m$ with $p < m$ for $0 \leq u \ll 1$. Let an arbitrary constant $R > 0$ be fixed. Suppose that

$$\Lambda_R := \{(k_1, k_2) \in \mathbb{N} \times \mathbb{N} : (k_1, k_2) \neq (0, 0), k_1(p-1) + k_2(m-1) \leq R \} \neq \emptyset.$$

Then as $\alpha \to 0$,

$$\lambda(\alpha) = \pi^2 + \sum_{(k_1, k_2) \in \Lambda_R} C_{k_1, k_2} \alpha^{k_1(p-1)+k_2(m-1)} + o(\alpha^R),$$

(1.4)

where

$$C_{1,0} = \frac{4}{\pi(p+1)} A_0^{(p-1)/q} \int_0^1 \frac{1-s^{p+1}}{(1-s^2)^{3/2}} ds,$$

(1.5)

$$C_{0,1} = \frac{4}{\pi(m+1)} A_0^{(m-1)/q} \int_0^1 \frac{1-s^{m+1}}{(1-s^2)^{3/2}} ds,$$

(1.6)

$$A_0 = \frac{\pi}{2T_0}, \quad T_0 := \int_0^1 \frac{s^q}{\sqrt{1-s^2}} ds,$$

(1.7)

and $C_{2,0}, C_{1,1}, \cdots$ are the constants obtained explicitly.

**Example.** Let $p = 2, m = 3$ and $R = 4$ in Theorem 1.1. Then by (1.4),

$$\lambda(\alpha) = \pi^2 + C_{1,0} \alpha + C_{2,0} \alpha^2 + C_{0,1} \alpha^2 + C_{3,0} \alpha^3 + C_{1,1} \alpha^3 + C_{4,0} \alpha^4 + C_{2,1} \alpha^4 + C_{0,2} \alpha^4 + o(\alpha^4).$$

(1.8)

This example shows how easy our method is to obtain the asymptotic formula for $\lambda_q(\alpha)$ if we are numerically supported.

The limiting case of Theorem 1.1 is the following Theorem 1.2.

**Theorem 1.2.** Assume that $f_0(u) = u^p + u^m$ ($1 < p < m$). Suppose that $g(u) = o(u^N)$ as $u \to 0$ for any positive integer $N$. Then as $\alpha \to 0$,

$$\lambda(\alpha) = \lambda_0(\alpha) + o(\alpha^N).$$

(1.9)
The typical example of \( g(u) \) in Theorem 1.2 is \( g(u) := e^{-1/u} \ (u > 0), \ g(0) := 0. \) Theorems 1.1 and 1.2 are obtained from the following Theorem 1.3. Let

\[
F_0(u) := \int_0^u f_0(s)ds, \quad G(u) := \int_0^u g(s)ds.
\]

**Theorem 1.3.**

(i) As \( \alpha \to 0, \)

\[
\sqrt{\lambda(\alpha)} = \pi + \sum_{k=1}^{\infty} D_k + E_0,
\]  

where

\[
D_k = 2 \int_0^1 \frac{(2k - 1)!!}{k!2^k} \frac{1}{\sqrt{1 - s^2}} H_\alpha(s)^k ds,
\]

\[
H_\alpha(s) = \frac{2 \lambda\|u_\alpha\|_\infty}{\lambda\|u_\alpha\|_\infty^2} \frac{F_0(\|u_\alpha\|_\infty) - F_0(\|u_\alpha\|_\infty s)}{1 - s^2},
\]

\[
L_\alpha(s) = \frac{2 \lambda\|u_\alpha\|_\infty}{\lambda\|u_\alpha\|_\infty^2} \frac{G(\|u_\alpha\|_\infty) - G(\|u_\alpha\|_\infty s)}{1 - s^2},
\]

\[
E_0 = 2 \int_0^1 \frac{1}{\sqrt{1 - s^2}}
\]

\[
\times \left[ \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k H_\alpha(s)^k - \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k (H_\alpha(s) + L_\alpha(s))^k \right]
\]

\[
\times \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} H_\alpha(s)^k \right] \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} (H_\alpha(s) + L_\alpha(s))^k \right] ds
\]

and \((2k - 1)!! = (2k - 1)(2k - 3) \cdots 3 \cdot 1\) for \( k \geq 1. \)

(ii) As \( \alpha \to 0, \)

\[
\sqrt{\lambda(\alpha)} = \frac{2\|u_\alpha\|_\infty}{\alpha^q} \left( T_0 + \sum_{k=1}^{\infty} T_k + E_1 \right),
\]  

where \( T_0 \) is the constant defined in (1.7) and

\[
T_k = \int_0^1 \frac{(2k - 1)!!}{k!2^k} \frac{s^q}{\sqrt{1 - s^2}} H_\alpha(s)^k ds \quad (k = 1, 2, \ldots),
\]

\[
E_1 = \int_0^1 \frac{s^q}{\sqrt{1 - s^2}}
\]

\[
\times \left[ \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k H_\alpha(s)^k - \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k (H_\alpha(s) + L_\alpha(s))^k \right]
\]

\[
\times \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} H_\alpha(s)^k \right] \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} (H_\alpha(s) + L_\alpha(s))^k \right] ds.
\]
Theorem 1.1 is the typical case to show that the scheme established in Theorem 1.3 is quite useful to compute the local bifurcation diagrams numerically.

The rest of this paper is organized as follows. We prove Theorem 1.3 in the next Section 2. In the proof, the method developed in [11] is useful, although it is more complicated than that of [11]. For Theorems 1.1 and 1.2, it is enough to prove Theorem 1.1, since the proof of Theorem 1.2 is variant of that of Theorem 1.1. We prove it in Section 3.

2 Proof of Theorem 1.3

In what follows, we always assume that $0 < \alpha \ll 1$. Further, $C$ denotes various positive constants independent of $\alpha$. We write $\lambda = \lambda(\alpha)$ for simplicity. Let $u_\alpha (\|u_\alpha\|_q = \alpha)$ be the solutions associated with $\lambda$. We begin with notations and the fundamental properties of $u_\alpha$. We know from [1] that for $\alpha > 0$,

\begin{equation}
\begin{aligned}
u_\alpha(t) &= u_\alpha(1 - t), \quad 0 < t \leq 1, \\
u_\alpha \left( \frac{1}{2} \right) &= \max_{0 \leq t \leq 1} u_\alpha(t) = \|u_\alpha\|_\infty, \\
u'_\alpha(t) &> 0, \quad 0 < t < \frac{1}{2}.
\end{aligned}
\end{equation}

Multiply (1.1) by $u'_\alpha(t)$. Then we obtain

\begin{equation}
[u''_\alpha(t) + \lambda u_\alpha(t) - f_0(u_\alpha(t)) - g(u_\alpha(t))]u'_\alpha(t) = 0.
\end{equation}

This implies that

\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} u'_\alpha(t)^2 + \frac{1}{2} \lambda u_\alpha(t)^2 - F_0(u_\alpha(t)) - G(u_\alpha(t)) \right) = 0.
\end{equation}

Namely, for $t \in \bar{I}$,

\begin{equation}
\begin{aligned}
\frac{1}{2} u'_\alpha(t)^2 + \frac{1}{2} \lambda u_\alpha(t)^2 - F_0(u_\alpha(t)) - G(u_\alpha(t)) &= \text{constant} \\
&= \frac{1}{2} \lambda \|u_\alpha\|_\infty^2 - F_0(\|u_\alpha\|_\infty) - G(\|u_\alpha\|_\infty).
\end{aligned}
\end{equation}

This implies that for $0 \leq t \leq 1/2$,

\begin{equation}
u'_\alpha(t) = \sqrt{A(u_\alpha(t)) - 2B(u_\alpha(t))},
\end{equation}

where

\begin{equation}
\begin{aligned}
A(\theta) &= \lambda(\|u_\alpha\|_\infty^2 - \theta^2) - 2(F_0(\|u_\alpha\|_\infty) - F_0(\theta)), \\
B(\theta) &= G(\|u_\alpha\|_\infty) - G(\theta).
\end{aligned}
\end{equation}
Note that, by (B.1), \( B(\theta) \geq 0 \) for \( 0 \leq \theta \leq \|u_\alpha\|_\infty \). By this and (2.7), \( A(\theta) \geq 0 \) for \( 0 \leq \theta \leq \|u_\alpha\|_\infty \).

**Proof of Theorem 1.3 (i).** We put \( s := u_\alpha(t)/\|u_\alpha\|_\infty \). Then by (2.1), (2.3) and (2.7),

\[
\frac{1}{2} = \int_0^{1/2} \frac{u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t)) - 2B(u_\alpha(t))}} \tag{2.10}
\]

\[
= \int_0^{1/2} \frac{u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))}} + \int_0^{1/2} \frac{u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t)) - 2B(u_\alpha(t))}} - \int_0^{1/2} \frac{u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))}} := I + II.
\]

We first calculate \( I \). By (A.1), for \( 0 \leq s \leq 1 \),

\[
H_\alpha(s) \leq \frac{2}{\lambda \|u_\alpha\|_\infty^2} f_0(\|u_\alpha\|_\infty) \|u_\alpha\|_\infty (1 - s) \leq C f_0(\|u_\alpha\|_\infty) \|u_\alpha\|_\infty \ll 1. \tag{2.11}
\]

Furthermore, for \( |x| \ll 1 \),

\[
(1 - x)^{-1/2} = 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} x^k. \tag{2.12}
\]

Put \( \theta = \|u_\alpha\|_\infty s \). By this, (2.10), Taylor expansion and Lebesgue’s convergence theorem,

\[
I = \int_0^{\|u_\alpha\|_\infty} \frac{1}{\sqrt{\lambda} \|u_\alpha\|_\infty} d\theta \tag{2.13}
\]

\[
= \int_0^{\|u_\alpha\|_\infty} \frac{1}{\sqrt{\lambda (\|u_\alpha\|_\infty^2 - \theta^2)} - 2(F_0(\|u_\alpha\|_\infty) - F_0(\theta))} d\theta
\]

\[
= \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{1}{\sqrt{1 - s^2}} (1 - H_\alpha(s))^{-1/2} ds
\]

\[
= \frac{1}{\sqrt{\lambda}} \left( \int_0^1 \frac{1}{\sqrt{1 - s^2}} ds + \int_0^1 \frac{1}{\sqrt{1 - s^2}} \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} H_\alpha(s)^k ds \right)
\]

\[
= \frac{1}{2\sqrt{\lambda}} \left( \pi + \sum_{k=1}^{\infty} D_k \right)
\].
We next calculate $II$. We see that

$$II = \int_0^{1/2} \frac{\sqrt{A(u_\alpha(t))} - \sqrt{A(u_\alpha(t)) - 2B(u_\alpha(t))}u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t)) - 2B(u_\alpha(t))}}$$

$$= \int_0^{\|u_\alpha\|\infty} \frac{\sqrt{A(\theta)} - \sqrt{A(\theta) - 2B(\theta)}d\theta}{\sqrt{A(\theta) - 2B(\theta)\sqrt{A(\theta)}}}$$

$$= \frac{1}{\sqrt{\lambda}} \int_0^{1} \frac{1}{\sqrt{1 - s^2}} \times (1 - H_\alpha(s))^{-1/2} (1 - H_\alpha(s) - L_\alpha(s))^{-1/2} ds$$

$$= \frac{1}{\sqrt{\lambda}} \int_0^{1} \frac{1}{\sqrt{1 - s^2}} \times \left[ \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k H_\alpha(s)^k - \sum_{k=1}^{\infty} \left( \frac{1/2}{k} \right) (-1)^k (H_\alpha(s) + L_\alpha(s))^k \right]$$

$$\times \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} H_\alpha(s)^k \right] \left[ 1 + \sum_{k=1}^{\infty} \frac{(2k - 1)!!}{k!2^k} (H_\alpha(s) + L_\alpha(s))^k \right] ds.$$

By this, (2.10) and (2.13), we obtain (1.10). Thus the proof is complete.

**Proof of Theorem 1.3 (ii).** By (2.1), (2.7) and putting $s := u_\alpha(t)/\|u_\alpha\|\infty$

$$\alpha^q = 2 \int_0^{1/2} u_\alpha(t)^q dt$$

$$= 2 \int_0^{1/2} \frac{u_\alpha(t)^q u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))} - 2B(u_\alpha(t))}$$

$$= 2 \int_0^{1/2} \frac{u_\alpha(t)^q u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))}}$$

$$+ \left( 2 \int_0^{1/2} \frac{u_\alpha(t)^q u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))} - 2B(u_\alpha(t))} - 2 \int_0^{1/2} \frac{u_\alpha(t)^q u'_\alpha(t)dt}{\sqrt{A(u_\alpha(t))}} \right).$$

Then by the same calculation as that in the proof of Theorem 1.3 (i), we obtain (1.15). Thus the proof is complete.

## 3 Proof of Theorem 1.1

The proof is divided into several steps.
**Step 1.** We know that for \( k = 1, 2, \ldots \)

\[
D_k = d_k \lambda^{-k} \| u_\alpha \|_{\infty}^{k(p-1)},
\]

(3.1)

\[
H_\alpha(s) = \frac{2}{\lambda(p+1)} \left\{ \frac{1-s^{p+1}}{1-s^2} \right\} \| u_\alpha \|_{p-1}^2,
\]

(3.2)

\[
L_\alpha(s) = \frac{2}{\lambda(m+1)} \left\{ \frac{1-s^{m+1}}{1-s^2} \right\} \| u_\alpha \|_{m-1}^m,
\]

(3.3)

where

\[
d_k = \frac{2(2k-1)!!}{k!(p+1)^k} \int_0^1 \frac{(1-s^{p+1})^k}{(1-s^2)^{k+1/2}} ds.
\]

(3.4)

To understand the scheme of the calculation, it is sufficient for us to consider the typical case where

\[ \Lambda_R = \{(1, 0), (2, 0), (0, 1), (1, 1)\}. \]

For example, let \( p = 2, m = 8/3, R = 8/3 \). Then \( \Lambda_R \) coincides with the set above. By (1.10), (1.14) and direct calculation, we obtain

\[
\sqrt{\lambda} = \pi + \frac{d_1}{\lambda} \| u_\alpha \|_{\infty}^{-(p-1)} + \frac{d_2}{\lambda^2} \| u_\alpha \|_{2(p-1)}^2 + \frac{J_1}{\lambda} \| u_\alpha \|_{m-1}^m
\]

\[
+ \frac{J_{1,1}}{\lambda^2} \| u_\alpha \|_{p-1+m-1} + o(\| u_\alpha \|_{p-1+m-1}^2),
\]

(3.5)

where

\[
J_1 = \frac{2}{m+1} \int_0^1 \frac{1-s^{m+1}}{(1-s^2)^{3/2}} ds,
\]

(3.6)

\[
J_{1,1} = \frac{6}{(p+1)(m+1)} \int_0^1 \frac{(1-s^{p+1})(1-s^{m+1})}{(1-s^2)^{5/2}} ds.
\]

(3.7)

By (1.15) and the same argument as that to obtain (3.5), we obtain

\[
\sqrt{\lambda} = \frac{2}{\alpha t} \| u_\alpha \|_{\infty}^q \left( T_0 + \sum_{k=1}^\infty \frac{t_k}{\lambda^k} \| u_\alpha \|_{\infty}^{k(p-1)} + \frac{K_1}{\lambda} \| u_\alpha \|_{\infty}^{m-1}
\]

\[
+ \sum_{k_1+k_2=2} K_{k_1,k_2} \| u_\alpha \|_{\infty}^{k_1(p-1)+k_2(m-1)} \right) + o(\| u_\alpha \|_{\infty}^{p-1+m-1}),
\]

(3.8)

where

\[
t_k = \frac{(2k-1)!!}{k!(p+1)^k} \int_0^1 \frac{s^q(1-s^{p+1})^k}{(1-s^2)^{k+1/2}} ds,
\]

(3.9)

\[
K_1 = \frac{1}{m+1} \int_0^1 \frac{s^q(1-s^{m+1})}{(1-s^2)^{3/2}} ds.
\]

(3.10)
and $K_{k_1,k_2}$ are the constant determined inductively.

**Step 2.** Recall that $A_0 = \pi/(2T_0)$ in (1.7). By (3.5) and (3.8),

$$\sqrt{\lambda} = \pi(1 + o(1)) = \frac{2\|u_\alpha\|_\infty^q}{\alpha^q}T_0(1 + o(1)). \quad (3.11)$$

This implies that

$$\|u_\alpha\|_\infty = A_0^{1/q} \alpha(1 + o(1)). \quad (3.12)$$

By this and (3.5)

$$\lambda = \left(\sqrt{\lambda}\right)^2 = \left(\pi + \frac{d_1}{\pi^2}A_0^{(p-1)/q} \alpha^{p-1}(1 + o(1))\right)^2 \quad (3.13)$$

$$= \pi^2 + 2\frac{d_1}{\pi}A_0^{(p-1)/q} \alpha^{p-1}(1 + o(1))$$

$$: = \pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1}).$$

**Step 3.** By (3.8),

$$\sqrt{\lambda} = \frac{2\|u_\alpha\|_\infty^q}{\alpha^q}(T_0 + \frac{t_1}{\lambda}\|u_\alpha\|_\infty^{p-1}(1 + o(1))). \quad (3.14)$$

By this, and (3.13),

$$\sqrt{\lambda} = \left(\pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1})\right)^{1/2} \quad (3.15)$$

$$= \frac{2\|u_\alpha\|_\infty^q}{\alpha^q}(T_0 + \frac{t_1}{\pi^2}\|u_\alpha\|_\infty^{p-1}(1 + o(1))).$$

By this, (3.11) and Taylor expansion,

$$\|u_\alpha\|_\infty = A_0^{1/q} \alpha \left(1 + \frac{C_{1,0}}{2q\pi^2} \alpha^{p-1} + o(\alpha^{p-1})\right) \quad (3.16)$$

$$\times \left(1 - \frac{t_1}{qT_0\pi^2}A_0^{(p-1)/q} \alpha^{p-1} + o(\alpha^{p-1})\right)$$

$$= A_0^{1/q} \alpha \left(1 + Q_1 \alpha^{p-1} + o(\alpha^{p-1})\right),$$

where

$$Q_1 = \frac{1}{q\pi^2} \left(\frac{C_{1,0}}{2} - \frac{t_1}{T_0}A_0^{(p-1)/q}\right).$$
By this, (3.5), (3.13), (3.14) and Taylor expansion,

\[ \sqrt{\lambda} = \pi + d_1 A_0^{(p-1)/q} \alpha^{p-1} \left( 1 + Q_1 \alpha^{p-1} + o(\alpha^{p-1}) \right) \]

\[ \times \left( \pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1}) \right)^{-1} \]

\[ + d_2 A_0^{2(p-1)/q} \alpha^{2(p-1)} \left( 1 + Q_1 \alpha^{p-1} + o(\alpha^{p-1}) \right)^{2(p-1)} \]

\[ \times \left( \pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1}) \right)^{-2} \]

\[ + J_1 A_0^{(m-1)/q} \alpha^{m-1} \left( 1 + Q_1 \alpha^{p-1} + o(\alpha^{p-1}) \right)^{m-1} \]

\[ \times \left( \pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1}) \right)^{-1} \]

\[ + J_{1,1} A_0^{(p-1)+(m-1)/q} \alpha^{p-1+m-1} \left( \pi^2 + C_{1,0} \alpha^{p-1} + o(\alpha^{p-1}) \right)^{-2} \]

\[ + o(\alpha^{p-1+m-1}) \]

\[ = \pi + \frac{d_1}{\pi^2} A_0^{(p-1)/q} \alpha^{p-1} \left( 1 + (p-1)Q_1 \alpha^{p-1} + o(\alpha^{p-1}) \right) \]

\[ \times \left( 1 - \frac{C_{1,0}}{\pi^2} \alpha^{p-1} + o(\alpha^{p-1}) \right) \]

\[ + \frac{d_2}{\pi^4} A_0^{2(p-1)/q} \alpha^{2(p-1)} + O(\alpha^{3(p-1)}) \]

\[ + \frac{J_1}{\pi^2} A_0^{(m-1)/q} \alpha^{m-1} \left( 1 + (m-1)Q_1 \alpha^{p-1} + o(\alpha^{p-1}) \right) \]

\[ \times \left( 1 - \frac{C_{1,0}}{\pi^2} \alpha^{p-1} + o(\alpha^{p-1}) \right) \]

\[ + \frac{J_{1,1}}{\pi^4} A_0^{(p-1+m-1)/q} \alpha^{p-1+m-1} + o(\alpha^{p-1+m-1}) \]

\[ = \pi + \frac{d_1}{\pi^2} A_0^{(p-1)/q} \alpha^{p-1} + \frac{J_1}{\pi^2} A_0^{(m-1)/q} \alpha^{m-1} + Q_2 \alpha^{2(p-1)} \]

\[ + Q_3 \alpha^{p-1+m-1} + o(\alpha^{p-1+m-1}), \]

where

\[ Q_2 = \frac{d_1}{\pi^2} A_0^{(p-1)/q} \left( (p-1)Q_1 - \frac{C_{1,0}}{\pi^2} \right) + \frac{d_2}{\pi^4} A_0^{2(p-1)/q}, \quad (3.18) \]

\[ Q_3 = \frac{J_1}{\pi^2} A_0^{(m-1)/q} \left( (m-1)Q_1 - \frac{C_{1,0}}{\pi^2} \right) + \frac{J_{1,1}}{\pi^4} A_0^{(p-1+m-1)/q}. \quad (3.19) \]

This implies

\[ \lambda = \pi^2 + C_{1,0} \alpha^{p-1} + C_{0,1} \alpha^{m-1} + C_{2,0} \alpha^{2(p-1)} + C_{1,1} \alpha^{p-1+m-1} \]

\[ + o(\alpha^{p-1+m-1}), \quad (3.20) \]

where

\[ C_{2,0} = \frac{C_{1,0}^2}{4\pi^2} + 2\pi Q_2, \quad C_{1,1} = \frac{C_{1,0} C_{0,1}}{2\pi^2} + 2\pi Q_3. \quad (3.21) \]
We note that, by the argument above, the remainder term \( o(\alpha^{p-1+m-1}) = O(\alpha^{3(p-1)}) + O(\alpha^{2(m-1)}) = o(\alpha^R) \). Indeed, \( (3,0),(0,2) \not\in \Lambda_R \) here, and by definition of \( \Lambda_R \), we have \( 3(p-1) > R \) and \( 2(m-1) > R \).

**Step 4.** By repeating the arguments above, we obtain Theorem 1.1. Thus the proof is complete.

## References


**Received:** February 4, 2013