A Simple Formula for a Generalized Conditional
Wiener Integral and its Applications

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Abstract

Let $C[0,t]$ denote a generalized Wiener space, the space of real-
valued continuous functions on the interval $[0,t]$ and define a random
vector $Z_n : C[0,t] \to \mathbb{R}^n$ by $Z_n(x) = (\int_0^{t_1} h(s)dx(s), \ldots, \int_0^{t_n} h(s)dx(s))$, where $0 < t_1 < \cdots < t_n = t$ is a partition of $[0,t]$ and $h \in L_2[0,t]$ with
$h \neq 0$ a.e. In this paper, with the conditioning function $Z_n$, we derive a
simple formula for a generalized conditional Wiener integral of functions
on $C[0,t]$. As applications of the formula, we evaluate the generalized
conditional Wiener integrals of various functions on $C[0,t]$.

Mathematics Subject Classification: 28C20

Keywords: Conditional Wiener integral, Simple formula for conditional
Wiener integral, Wiener measure

1 Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, the space of continuous real-valued functions $x$ on $[0,t]$ with $x(0) = 0$. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by Fourier-transform [10]. But Yeh’s inversion formula is very complicated in its applications when the conditioning function is vector-valued. In [5], Park and Skoug derived a simple formula for conditional Wiener integrals on $C_0[0,t]$ with the vector-valued
conditioning function \( X : C_0[0, t] \rightarrow \mathbb{R}^n \) given by \( X(x) = (x(t_1), \ldots, x(t_n)) \) where \( \tau : 0 = t_0 < t_1 < \cdots < t_n = t \) is a partition of the interval \([0, t]\). In their simple formula, they expressed the conditional Wiener integral directly in terms of ordinary Wiener integral.

On the other hand, let \( C[0, t] \) denote the space of continuous real-valued functions on the interval \([0, t]\). Im and Ryu introduced a probability measure \( w_\varphi \) on \( C[0, t] \), where \( \varphi \) is a probability measure on the Borel class of \( \mathbb{R} \) \([4, 7]\). When \( \varphi = \delta_0 \), the Dirac measure concentrated at 0, \( w_\varphi \) is exactly the Wiener measure on \( C_0[0, t] \). On the space \( C[0, t] \), the author derived a simple formula for the conditional Wiener \( w_\varphi \)-integral of functions on \( C[0, t] \) with the vector-valued conditioning function \( X_\tau : C[0, t] \rightarrow \mathbb{R}^{n+1} \) given by \( X_\tau(x) = (x(t_0), x(t_1), \ldots, x(t_n)) \), which generalized the Park and Skoug’s formula \([2]\). Using the formula with the conditioning \( X_\tau \), he evaluated the conditional Wiener \( w_\varphi \)-integral of function of the form \( F_m(x) = \int_0^1 (x(s))^m ds \) for any positive integer \( m \) \([2, 3]\). Let \( h \in L_2[0, t] \) with \( h \neq 0 \) a.e. on \([0, t]\). Define a stochastic process \( Z : C[0, t] \times [0, t] \rightarrow \mathbb{R} \) by \( Z(x, s) = \int_0^s h(u) dx(u) \) for \( x \in C[0, t] \) and \( s \in [0, t] \), where the integral denotes the Paley-Wiener-Zygmund integral, and let

\[
Z_n(x) = (Z(x, t_1), \ldots, Z(x, t_n)).
\]

In this paper, with the conditioning function \( Z_n \), we derive a simple formula for a generalized conditional Wiener \( w_\varphi \)-integral of functions on \( C[0, t] \), which generalizes the Park and Skoug’s formula and the formula in \([2]\) when \( \varphi = \delta_0 \). As applications of the formula, we evaluate the generalized conditional Wiener \( w_\varphi \)-integrals of functions of the forms

\[
F_m(Z(x, \cdot)), \ Z(x, s_1)Z(x, s_2) \quad \text{and} \quad \int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v)
\]

for \( x \in C[0, t] \), where \( 0 < s_1 < s_2 \leq t \) and \( \sigma \) is a complex Borel measure on \( L_2[0, t] \). Note that these functions are interested in the Feynman integration theories and quantum mechanics.

Let \((C[0, t], \mathcal{B}(C[0, t]), w_\varphi)\) be the analogue of Wiener space associated with a probability measure \( \varphi \) on the Borel class of \( \mathbb{R} \), where \( \mathcal{B}(C[0, t]) \) denotes the Borel class of \( C[0, t] \) \([4, 7, 8]\). Let \( \{d_j : j = 1, 2, \ldots\} \) be a complete orthonormal subset of \( L_2[0, t] \) such that each \( d_j \) is of bounded variation. For \( v \in L_2[0, t] \) and \( x \in C[0, t] \), let \( \langle v, x \rangle = \int_0^t v(s) dx(s) = \lim_{n \to \infty} \sum_{j=1}^n \langle v, d_j \rangle \int_0^t d_j(s) dx(s) \) if the limit exists, where \( \langle \cdot, \cdot \rangle \) denotes the inner product over \( L_2[0, t] \). \( \langle v, x \rangle \) is called the Paley-Wiener-Zygmund integral of \( v \) according to \( x \).

Applying Theorem 3.5 in \([4]\), we can easily prove the following theorem.

**Theorem 1.1** Let \( \{v_1, \ldots, v_r\} \) be an orthonormal subset of \( L_2[0, t] \). Then \( (v_1, \cdot), \ldots, (v_r, \cdot) \) are independent random variables on \( C[0, t] \) and each \( (v_j, \cdot) \)
has the standard normal distribution. Moreover, if \( f : \mathbb{R}^r \to \mathbb{R} \) is Borel measurable, then
\[
\int_C f((v_1, x), \ldots, (v_r, x)) dw_c(x)
= \left( \frac{1}{2\pi} \right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(u_1, u_2, \ldots, u_r) \exp \left\{ -\frac{1}{2} \sum_{j=1}^r u_j^2 \right\} du_1, u_2, \ldots, u_r,
\]
where \( * \) means that if either side exists then both sides exist and they are equal.

2 A simple formula for a generalized conditional Wiener integral

In this section, we derive a simple formula for a generalized conditional Wiener \( w_c \)-integral on \( C[0, t] \) with the conditioning function \( Z_n \) as given in Section 1.

Let \( h \in L_2[0, t] \) with \( h \neq 0 \) a.e. on \([0, t] \). Define a stochastic process \( Z : C[0, t] \times [0, t] \to \mathbb{R} \) by \( Z(x, s) = \int_0^s h(u) dx(u) \) for \( x \in C[0, t] \) and \( s \in [0, t] \). Let \( b(s) = \int_0^s h^2(u) du \) for \( 0 \leq s \leq t \). We note that \( Z \) is not the process \( z \) given in [6] since \( \hat{Z} \) is defined on \( C[0, t] \), but \( z \) is defined on the Wiener space \( C_0[0, t] \).

Let \( 0 = t_0 < t_1 < \cdots < t_n = t \) be a partition of \([0, t] \) and define a random vector \( Z_n : C[0, t] \to \mathbb{R}^n \) by \( Z_n(x) = (Z(x, t_1), \ldots, Z(x, t_n)) \) for \( x \in C[0, t] \).

For any \( x \) in \( C[0, t] \), define the polygonal function \( [Z(x, \cdot)] \) of \( x \) by
\[
[Z(x, \cdot)](s) = \sum_{j=1}^n \chi(t_{j-1}, t_j)(s) b(t_j) Z(x, t_{j-1}) + \chi(t_{j-1}, t_j)(s) b(t_j) Z(x, t_j)
\]
(1)

for \( s \in [0, t] \), where \( \chi(t_{j-1}, t_j) \) denotes the indicator function. Similarly, for \( \vec{\xi}_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \), define the polygonal function \( [\vec{\xi}_n] \) of \( \vec{\xi}_n \) by (1), where \( Z(x, t_j) \) is replaced by \( \xi_j \) for \( j = 1, \ldots, n \) and \( \xi_0 = 0 \). Note that \( Z(x, 0) = [Z(x, \cdot)](0) = 0 \) although \( x(0) \neq 0 \).

Theorem 2.1 Let \( \{v_1, v_2\} \) be an independent subset of \( L_2[0, t] \). Then \( \{(v_1, \cdot), (v_2, \cdot)\} \) are multivariate normally distributed on \( C[0, t] \) with mean vector \( \vec{0} \) and covariance matrix \( A = [(v_1, v_2)]_{2\times2} \).

Proof. We use the Gram-Schmidt orthonormalization process for \( \{v_1, v_2\} \). Let \( e_1 = \frac{1}{\|v_1\|} v_1, e_2 = \frac{1}{\|v_2 - (v_2, e_1) e_1\|} (v_2 - (v_2, e_1) e_1) \) and \( B \) be a Borel subset of \( \mathbb{R}^2 \). Then
\[
w_c(((v_1, \cdot), (v_2, \cdot)) \in B) = \int_{C[0, t]} \chi_B((v_1, x), (v_2, x)) dw_c(x)
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_B(\|v_1\|u_1, \langle v_2, e_1 \rangle u_1 + \|v_2 - \langle v_2, e_1 \rangle e_1\|u_2) \exp \left\{ -\frac{1}{2}(u_1^2 + u_2^2) \right\} du_1 du_2
\]

by Theorem 1.1, where \( \chi_B \) denotes the indicator function of \( B \). Let \( z_1 = \|v_1\|u_1 \) and \( z_2 = \langle v_2, e_1 \rangle u_1 + \|v_2 - \langle v_2, e_1 \rangle e_1\|u_2 \). Since \( \|v_1\|^2\|v_2 - \langle v_2, e_1 \rangle e_1\|^2 = |A| \neq 0 \) by the Schwarz’s inequality, the Jacobian determinant is given by

\[
\left| \begin{array}{cc}
\frac{1}{\|v_1\|v_2 - \langle v_2, e_1 \rangle e_1 \|u_1\|} & 0 \\
\frac{1}{\|v_1\|} & 1
\end{array} \right| = \frac{1}{\|v_1\|\|v_2 - \langle v_2, e_1 \rangle e_1\|} = |A|^{-1/2}
\]

so that by the change of variable theorem

\[
w_c((v_1, \cdot), (v_2, \cdot)) \in B)
\]

\[
\frac{1}{2\pi|A|^{1/2}} \int_B \exp \left\{ -\frac{1}{2} \left[ \frac{1}{\|v_1\|^2} z_1^2 + \frac{\|v_1\|^2}{|A|} \right] \left( -\langle v_2, e_1 \rangle z_1 + z_2 \right) \right\} dz_1 dz_2
\]

\[
\frac{1}{2\pi|A|^{1/2}} \int_B \exp \left\{ -\frac{1}{2} \left[ -\frac{\|v_2\|^2}{|A|} z_1^2 - 2\langle v_1, v_2 \rangle z_1 z_2 + \|v_1\|^2 z_2^2 \right] \right\} dz_1 dz_2
\]

\[
\frac{1}{2\pi|A|^{1/2}} \int_B \exp \left\{ -\frac{1}{2} \bar{z} A^{-1} \bar{z}^T \right\} d\bar{z}
\]

where \( \bar{z} = (z_1, z_2) \) and \( \bar{z}^T \) is the transpose of \( \bar{z} \), which completes the proof. \( \square \)

**Corollary 2.2** For \( s \in (0, t] \), \( Z(\cdot, s) \) is normally distributed with mean zero and variance \( b(s) \). Moreover, \( Z \) is a Gaussian process and its covariance function is given by

\[
E[Z(\cdot, s_1)Z(\cdot, s_2)] = b(\min\{s_1, s_2\}) \text{ for } s_1, s_2 \in [0, t].
\]

**Proof.** Let \( v_s = \chi_{[0,s]}h \). By Theorem 1.1, \( (v_s, \cdot) \) is normally distributed with mean zero and variance \( \|v_s\|^2 \). Since \( Z(x, s) = (v_s, x) \) for \( x \in C[0, t] \) and \( \|v_s\|^2 = b(s) \), we have the first part of the theorem. Since if \( s_1 = 0 \) or \( s_2 = 0 \) the proof of the second part of the theorem is trivial, it suffices to prove the second part of the theorem for the case \( 0 < s_1, s_2 \leq t \). Let \( v_1 = \chi_{[0,s_1]}h \) and \( v_2 = \chi_{[0,s_2]}h \). Since \( h \neq 0 \) a.e. on \( [0, t] \), \( v_1 \) and \( v_2 \) are independent vectors in \( L_2[0, t] \) so that by Theorem 2.1

\[
E[Z(\cdot, s_1)Z(\cdot, s_2)] = \text{Cov}((v_1, \cdot), (v_2, \cdot)) = \langle v_1, v_2 \rangle = \int_0^{\min\{s_1, s_2\}} h^2(u)du,
\]

which completes the proof. \( \square \)

**Remark 2.3** 1. We can Corollary 2.2 using direct calculation with an aid of Theorem 2.1, but the proof is tedious.
2. If \( h = 1 \) a.e. on \([0, t]\), then \( Z(x, s) = x(s) - x(0) \) for \( x \in C[0, t] \) and \( b(s) = s \). This means that \( Z \) is not the process \( z(x, s) = x(s)(x \in C_0[0, t]) \) in \([6] \), but they have same distribution.

3. If \( \varphi = \delta_0 \), the Dirac measure concentrated on 0, then \( w_{\delta_0} \) is concentrated on \( C_0[0, t] \) so that we can regard \( Z \) as the process on \( C_0[0, t] \times [0, t] \).

**Theorem 2.4** Let \( \{v_1, v_2\} \) be an independent subset of \( L_2[0, t] \). Then \( (v_2, \cdot) - (v_1, \cdot) \) is normally distributed with mean zero and variance \( \|v_2 - v_1\|^2 \).

**Proof.** Let \( B \) be a Borel subset of \( \mathbb{R} \). Then by Theorem 2.1

\[
\begin{align*}
w_{\varphi}((v_2, \cdot) - (v_1, \cdot) \in B) &= \frac{1}{2\pi |A|^2} \int_{\mathbb{R}^2} \chi_B(z_2 - z_1) \exp \left\{ -\frac{1}{2|A|} (\|v_2\|^2 z_1^2 - 2\langle v_1, v_2 \rangle z_1 z_2 + \|v_1\|^2 z_2^2) \right\} dz_1 dz_2.
\end{align*}
\]

Let \( u = z_2 - z_1 \). By the change of variable theorem

\[
\begin{align*}
w_{\varphi}((v_2, \cdot) - (v_1, \cdot) \in B) &= \frac{1}{2\pi |A|^2} \int_{\mathbb{R}^2} \chi_B(u) \exp \left\{ -\frac{1}{2|A|} (\|v_1 - v_2\|^2 z_1^2 - 2u(\langle v_1, v_2 \rangle - \|v_1\|^2) z_1 
\right. \\
&\quad \left. + \|v_1\|^2 u^2 \right\} dz_1 du \\
&= \frac{1}{\sqrt{2\pi \|v_2 - v_1\|}} \int_B \exp \left\{ \frac{1}{2|A|} \left( \frac{\langle v_1, v_2 \rangle - \|v_1\|^2}{\|v_2 - v_1\|} - \|v_1\|^2 \right) u^2 \right\} du \\
&= \frac{1}{\sqrt{2\pi \|v_2 - v_1\|}} \int_B \exp \left\{ -\frac{u^2}{2\|v_2 - v_1\|^2} \right\} du,
\end{align*}
\]

which completes the proof. \( \square \)

**Corollary 2.5** Let \( 0 \leq s_1 < s_2 \leq t \). Then \( Z(\cdot, s_2) - Z(\cdot, s_1) \) is normally distributed with mean zero and variance \( b(s_2) - b(s_1) \).

**Proof.** Suppose that \( s_1 = 0 \). Let \( v_2 = \chi_{[0, s_2]}h \). Then \( Z(\cdot, s_2) - Z(\cdot, s_1) = (v_2, \cdot) \) so that by Theorem 1.1, \( Z(\cdot, s_2) - Z(\cdot, s_1) \) is normally distributed with mean zero and variance \( \|v_1\|^2 = b(s_2) - b(s_1) \). Suppose that \( 0 < s_1 < s_2 \leq t \). Let \( v_1 = \chi_{[0, s_1]}h \). Then \( Z(\cdot, s_2) - Z(\cdot, s_1) = (v_2, \cdot) - (v_1, \cdot) \) which is normally distributed with mean zero and variance \( \|v_2 - v_1\|^2 \) by Theorem 2.4. Since \( \|v_2 - v_1\|^2 = b(s_2) - b(s_1) \), we have the corollary. \( \square \)

**Corollary 2.6** Let \( 0 \leq s_1 < s_2 < s_3 < s_4 \leq t \). Then \( Z(\cdot, s_2) - Z(\cdot, s_1) \) and \( Z(\cdot, s_4) - Z(\cdot, s_3) \) are independent.
Proof. By Corollary 2.2

\[ E[(Z(\cdot, s_4) - (Z(\cdot, s_3))(Z(\cdot, s_2) - (Z(\cdot, s_1))] \\
= b(s_2) - b(s_1) - [b(s_2) - b(s_1)] = 0 \\
= E[(Z(\cdot, s_4) - (Z(\cdot, s_3))E[(Z(\cdot, s_2) - (Z(\cdot, s_1))]. \]

Since \( Z(\cdot, s_2) - Z(\cdot, s_1) \) and \( Z(\cdot, s_4) - Z(\cdot, s_3) \) are normally distributed by Corollary 2.5, we have the corollary. \( \square \)

Corollary 2.7 \( Z \) is a generalized Brownian motion process.

Corollary 2.8 Let \( 0 = s_0 < s_1 < \cdots < s_m \leq t \). Then the random vector \( (Z(\cdot, s_1), \cdots, Z(\cdot, s_m)) \) has a joint density function \( W_m \) given by

\[ W_m(s_1, \cdots, s_m; u_1, \cdots, u_m) = \prod_{j=1}^{m} \frac{1}{2\pi(b(s_j) - b(s_{j-1}))} \exp \left\{ -\sum_{j=1}^{m} \frac{(u_j - u_{j-1})^2}{2(b(s_j) - b(s_{j-1}))} \right\} \]

where \( u_0 = 0 \) and \((u_1, \cdots, u_m) \in \mathbb{R}^m\).

Now, by Corollary 2.7 and Theorem 17.3 of [9], we have the following corollary.

Theorem 2.9 Let \( t_{j-1} < s < t_j \) for some \( j \in \{1, \cdots, n\} \). Then \( Z(\cdot, s) - [Z(\cdot, \cdot)](s) \) is normally distributed with mean zero and variance \( (b(t_j) - b(s))(b(s) - b(t_{j-1}))/b(t_j) - b(t_{j-1})) \).

Proof. For \( x \in C[0, t] \)

\[ Z(x, s) - [Z(x, \cdot)](s) = \alpha(Z(x, s) - Z(x, t_{j-1})) - \beta(Z(x, t_j) - Z(x, s)) \]

where \( \alpha = \frac{b(s) - b(t_j)}{b(t_j) - b(t_{j-1})} \) and \( \beta = \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} \). Let \( B \) be a Borel subset of \( \mathbb{R} \) and \( \Gamma = \frac{1}{(2\pi)^2(b(t_j) - b(s))(b(s) - b(t_{j-1}))} \frac{1}{2} \). Suppose that \( t_{j-1} > 0 \). By Corollary 2.8

\[ w_{\psi}(Z(\cdot, s) - [Z(\cdot, \cdot)](s) \in B) \]

\[ = \Gamma \left[ \frac{1}{2\pi b(s)} \int_{\mathbb{R}^3} \chi_B(\alpha(u_2 - u_1) - \beta(u_3 - u_2))W_3(t_{j-1}, s, t_j; u_1, u_2, u_3) \right] d(u_1, u_2, u_3). \]

Let \( z_1 = \alpha(u_2 - u_1) \) and \( z_2 = -\beta(u_3 - u_2) \). Then we have by the change of variable theorem

\[ w_{\psi}(Z(\cdot, s) - [Z(\cdot, \cdot)](s) \in B) \]
\[ \frac{\Gamma}{\alpha\beta} \int_{\mathbb{R}^2} \chi_B(z_1 + z_2) \exp\left\{ -\frac{z_1^2}{2\alpha^2(b(s) - b(t_{j-1}))} - \frac{z_2^2}{2\beta^2(b(t_j) - b(s))} \right\} \]

Using the similar method as used in the proof of Lemma 2.3 of [2] with Corollary 2.8, we can prove the theorem. \(\square\)

**Theorem 2.10** The process \(\{Z(\cdot, s) - [Z(\cdot, \cdot)](s) : 0 \leq s \leq t\}\) and \(Z_n\) are stochastically independent.

**Proof.** Suppose that \(s = 0\) and let \(B_1, B_2\) be two Borel subsets of \(\mathbb{R}\). Then

\[ w_\varphi(Z(\cdot, 0) - [Z(\cdot, \cdot)](0) \in B_1, Z(\cdot, t_j) \in B_2) \]

\[ = w_\varphi(Z(\cdot, 0) - [Z(\cdot, \cdot)](0) \in B_1)w_\varphi(Z(\cdot, t_j) \in B_2) \]

which shows that the independence of \(Z(\cdot, 0) - [Z(\cdot, \cdot)](0)\) and \(Z(\cdot, t_j)\). Let \(0 < s \leq t\) and \(s \in [t_{l-1}, t_l]\). By Corollary 2.2

\[ E[(Z(\cdot, s) - [Z(\cdot, \cdot)](s)Z(\cdot, t_j)] \]

\[ = b(\min\{s, t_j\}) - b(\min\{t_{l-1}, t_j\}) - \frac{b(s) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} b(\min\{t_l, t_j\}) \]

\[ = \begin{cases} b(s) - b(t_{l-1}) - \frac{b(s) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} (b(t_l) - b(t_{l-1})) & (l \leq j) \\ b(t_j) - b(t_j) - \frac{b(s) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} (b(t_j) - b(t_j)) & (j < l) \end{cases} \]

\[ = 0. \]

By Corollary 2.2 and Theorem 2.9, we have the theorem. \(\square\)

**Theorem 2.11** The processes \(\{Z(\cdot, s) - [Z(\cdot, \cdot)](s) : t_{j-1} \leq s \leq t_j\}\), where \(j = 1, \cdots, n\), are stochastically independent.

**Proof.** Let \(s_1 \in [t_{l-1}, t_l]\) and \(s_2 \in [t_{j-1}, t_j]\) with \(l < j\). If \(s_1 \in \{0, t_1, \cdots, t_n\}\) or \(s_1 \in \{0, t_1, \cdots, t_n\}\), then \(Z(\cdot, s_1) - [Z(\cdot, \cdot)](s_1)\), \(Z(\cdot, s_2) - [Z(\cdot, \cdot)](s_2)\) are independent using the same the method in the first part of the proof in Theorem 2.10. Suppose that \(t_{l-1} < s_1 < t_l < t_{j-1} < s_2 < t_j\). By Corollaries 2.2 and 2.6

\[ E[(Z(\cdot, s_1) - [Z(\cdot, \cdot)](s_1))(Z(\cdot, s_2) - [Z(\cdot, \cdot)](s_2))] \]

\[ = b(s_1) - b(s_1) - \frac{b(s_2) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (b(s_1) - b(s_1)) - b(t_{l-1}) - \frac{b(s_1) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} \]

\[ \times (b(t_l) - b(t_{l-1})) + b(t_{l-1}) + \frac{b(s_2) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (b(t_{l-1}) - b(t_{l-1})) + \]

\[ \frac{b(s_1) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} (b(t_l) - b(t_{l-1})) + \frac{b(s_1) - b(t_{l-1})}{b(t_l) - b(t_{l-1})} b(s_2) - b(t_{j-1}) \]

\[ = 0 \]
which completes the proof by Theorem 2.9. □

Applying the same method as used in the proof of Theorem 2 in [5, p.383] with Problem 4 of [1, p.216], we have the following theorem from Theorem 2.10.

**Theorem 2.12** Let \( F : C[0, t] \to \mathbb{C} \) be a function and \( F(Z(x, \cdot)) \) be integrable over the variable \( x \). Then for a Borel subset \( B \) of \( \mathbb{R}^n \)

\[
\int_{Z_n^{-1}(B)} F(Z(x, \cdot))dw_\varphi(x) = \int_B E[F(Z(x, \cdot)) - [Z(x, \cdot)] + [\xi_n]_B]dP_n(\xi_n)
\]

where \( P_n \) is the probability distribution of \( Z_n \) on \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \). Moreover, for \( P_n \)-a.e. \( \xi_n \in \mathbb{R}^n \) (for a.e. \( \xi_n \in \mathbb{R}^n \))

\[
E[F(Z(x, \cdot))|Z_n](\xi_n) = E[F(Z(x, \cdot)) - [Z(x, \cdot)] + [\xi_n]_B].
\]

**Remark 2.13**

1. \( Z_n \) is defined on \( C[0, t] \). This means that Theorem 2.12 is not a generalization of Theorem 2.9 in [2] since the conditioning function in Theorem 2.9 of [2] contains the point \( x(0) \) and \( x(0) \) has the probability distribution on \( (\mathbb{R}, \mathcal{B}(\mathbb{R}), \varphi) \).

2. If \( \varphi = \delta_0 \), the Dirac measure concentrated at 0, then \( w_\varphi \) is the Wiener measure on \( C_0[0, t] \) so that Theorem 2.12 is a generalization of Theorem 3 in [6].

3. If \( h = 1 \) a.e. on \( [0, t] \), then \( F(Z(x, \cdot)) = F(x - x(0)) \) and \( Z_n(x) = (x(t_1) - x(0), \ldots, x(t_n) - x(0)) \) so that Theorem 2.12 is a generalization of Theorem 3 in [5].

4. Theorem 2.12 is a generalization of Theorem 2.9 in [2] only when \( \varphi = \delta_0 \).

5. All the results of this section are independent of the choice of \( \varphi \).

## 3 Applications of the simple formula

In this section, we evaluate the conditional Wiener \( w_\varphi \)-integral of various functions on \( C[0, t] \) using Theorem 2.12.

**Theorem 3.1** Let \( F_m(x) = \int_0^t (x(s))^m ds \) (\( m \in \mathbb{N} \)) for \( x \in C[0, t] \). Then \( F_m(Z_n(x, \cdot)) \) is (Wiener) \( w_\varphi \)-integrable. Moreover, \( E[F_m(Z(x, \cdot))|Z_n](\xi_n) \) exists for a.e. \( \xi_n \in \mathbb{R}^n \) (hence \( P_n \)-a.e. \( \xi_n \in \mathbb{R}^n \)) and it is given by

\[
E[F_m(Z(x, \cdot))|Z_n](\xi_n) = \sum_{j=1}^{n} \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2k!(m-2k)!} \int_{t_{j-1}}^{t_j} \left( [\xi_n]_B(s) \right)^{m-2k} \\
\times \left( \frac{(b(t_j) - b(s))(b(s) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})} \right)^k ds
\]
where \([\cdot]\) denotes the greatest integer function.

**Proof.** By Corollary 2.2

\[
\int_{C[0,t]} |F_m(Z(x, \cdot))|dw_\varphi(x) \leq \int_0^t \int_{C[0,t]} |Z(x, s)|^m dw_\varphi(x)ds = \int_0^t \left[ \frac{1}{2\pi b(s)} \right]^{\frac{1}{2}} \int_\mathbb{R} |u|^m \exp \left\{ -\frac{u^2}{2b(s)} \right\} du ds.
\]

Let \(\alpha = \frac{u^2}{2b(s)}\). By the change of variable theorem

\[
\int_{C[0,t]} |F_m(Z(x, \cdot))|dw_\varphi(x) \leq \left( \frac{1}{\pi} \right)^{\frac{1}{2}} \int_{(0,t]} (2b(s))^{\frac{m}{2}} \int_0^\infty |\alpha|^{\frac{m+1}{2}-1} \exp \{-\alpha\} d\alpha ds = \Gamma\left( \frac{m+1}{2} \right) \left( \frac{1}{\pi} \right)^{\frac{1}{2}} \int_{(0,t]} (2b(s))^{\frac{m}{2}} ds < \infty
\]

since \(b\) is increasing, where \(\Gamma\) is the gamma function. For a.e. \(\vec{\xi}_n \in \mathbb{R}^n\) (hence for \(P_{Z_n}\)-a.e. \(\vec{\xi}_n \in \mathbb{R}^n\)), we have by Theorem 2.12

\[
E[F_m(Z(x, \cdot))|Z_n](\vec{\xi}_n) = \int_0^t \int_{C[0,t]} (Z(x, s) - [Z(x, \cdot)](s) + [\vec{\xi}_n]_b(s))^m dw_\varphi(x)ds.
\]

Using the same method as used in the proof of Theorem 3.1 in [5], we have the theorem by Theorem 2.9. \(\square\)

**Example 3.2** Let \(h = 1\) a.e. on \([0, t]\) and \(F_m\) be as given in Theorem 3.1. By Theorem 3.1, we have for a.e. \((\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\) (hence for \(P_{Z_n}\)-a.e. \(\vec{\xi}_n \in \mathbb{R}^n\))

\[
E[F_m(x - x(0))|(x(t_1) - x(0), \ldots, x(t_n) - x(0)) = (\xi_1, \ldots, \xi_n)] = \sum_{j=1}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{2^k k!(m - 2k)!} \int_{t_{j-1}}^{t_j} \left( \xi_{j-1} + \frac{s - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}) \right)^{m-2k} \left( \frac{(t_j - s)(s - t_{j-1})}{t_j - t_{j-1}} \right)^k ds
\]

\[
= \sum_{j=1}^n \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{l=0}^{m-2k} \frac{m!(l + k)!(t_j - t_{j-1})^{k+1}(\xi_{j-1} - \xi_{j-1})} {2^k!(m - 2k - l)(l + 2k + 1)!} ds
\]

using similar process as used in the proof of Theorem 3.1 in [3], where \(\xi = 0\).
Example 3.3 Let $F_m$ be as given in Theorem 3.1. By Theorem 3.1, we have for $m = 1$ and for a.e. $\xi_n \in \mathbb{R}^n$ (hence for $P_{Z_n}$-a.e. $\xi_n \in \mathbb{R}^n$)

\[
E[F_1(Z(x, \cdot))|Z_n](\xi_n)
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left( \xi_{j-1} + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (\xi_j - \xi_{j-1}) \right) ds
\]

\[
= \sum_{j=1}^{n} \left[ \left( \xi_{j-1} - \frac{b(t_{j-1})(\xi_j - \xi_{j-1})}{b(t_j) - b(t_{j-1})} \right)(t_j - t_{j-1}) + \frac{\xi_j - \xi_{j-1}}{b(t_j) - b(t_{j-1})} (B(t_j) - B(t_{j-1})) \right],
\]

where $\frac{d}{ds}B(s) = b(s)$. Furthermore, we have for a.e. $\xi_n \in \mathbb{R}^n$ (hence for $P_{Z_n}$-a.e. $\xi_n \in \mathbb{R}^n$)

\[
E[F_2(Z(x, \cdot))|Z_n](\xi_n) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[ \left( \xi_{j-1} + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (\xi_j - \xi_{j-1}) \right)^2 \right.
+ \frac{(b(t_j) - b(s))(b(s) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})} \bigg] ds
\]

and

\[
E[F_3(Z(x, \cdot))|Z_n](\xi_n)
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[ \left( \xi_{j-1} + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (\xi_j - \xi_{j-1}) \right)^3 \right.
+ 3 \left( \xi_{j-1} + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (\xi_j - \xi_{j-1}) \right) (b(t_j) - b(s))(b(s) - b(t_{j-1})) \bigg] ds.
\]

Theorem 3.4 Let $0 < s_1 < s_2 \leq t$ and let $s_1 \in [t_{j-1} - t_j], s_2 \in [t_{j-1} - t_j]$. For $x \in C[0, t]$, let $G(x) = x(s_1)x(s_2)$. Then $G(Z(x, \cdot))$ is $w_{\phi}$-integrable. Moreover, $E[G(Z(x, \cdot))|Z_n](\xi_n)$ exists for a.e. $\xi_n \in \mathbb{R}^n$ (hence $P_{Z_n}$-a.e. $\xi_n \in \mathbb{R}^n$) and it is given by

\[
E[G(Z(x, \cdot))|Z_n](\xi_n) = \begin{cases} 
\frac{[\xi_n]_b(s_1)}{b(t_j) - b(t_{j-1})} [\xi_n]_b(s_2) + [\xi_n]_b(s_1)[\xi_n]_b(s_2) & (l \neq j) \\
\frac{[\xi_n]_b(s_1)}{b(t_j) - b(t_{j-1})} [\xi_n]_b(s_2) & (l = j) 
\end{cases}
\]

Proof. By Corollary 2.8 and the change of variable theorem

\[
\int_{C[0,t]} |G(Z(x, \cdot))| dw_{\phi}(x)
= \left[ \frac{1}{(2\pi)^2 b(s_1)(b(s_2) - b(s_1))} \right]^\frac{1}{2} \int_{\mathbb{R}^2} |u_1||u_1 + u_2| \exp \left\{ -\frac{u_1^2}{2b(s_1)} \right\}
- \frac{u_2^2}{2(b(s_2) - b(s_1))} \right\} d(u_1, u_2)
\]
\begin{align*}
\leq b(s_1) + \frac{2}{\pi} \sqrt{b(s_1)(b(s_2) - b(s_1))} < \infty.
\end{align*}

For a.e. $\vec{\xi}_n \in \mathbb{R}^n$ (hence for $P_{Z_n}$-a.e. $\vec{\xi}_n \in \mathbb{R}^n$), we have by Theorems 2.9 and 2.12

\[ E[G(Z(x, \cdot))|Z_n](\vec{\xi}_n) = \int_{C[0,t]} (Z(x, s_1) - [Z(x, \cdot)](s_1))(Z(x, s_2) - [Z(x, \cdot)](s_2))dw_\varphi(x) + [\vec{\xi}_n]_b(s_1) \times [\vec{\xi}_n]_b(s_2). \]

Suppose that $l \neq j$. Then by Theorems 2.9 and 2.11

\[ E[G(Z(x, \cdot))|Z_n](\vec{\xi}_n) = [\vec{\xi}_n]_b(s_1)[\vec{\xi}_n]_b(s_2). \]

Now suppose that $l = j$. Then by Corollary 2.2

\[ E[G(Z(x, \cdot))|Z_n](\vec{\xi}_n) = (b(s_1) - b(t_{j-1}) - \frac{b(s_2) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(b(s_1) - b(t_{j-1})) - b(t_{j-1}) - \frac{b(s_1) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(b(s_2) - b(t_{j-1})) + b(t_{j-1}) + \frac{b(s_2) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} \times \frac{b(s_1) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(b(t_j) - b(t_{j-1})) + [\vec{\xi}_n]_b(s_1)[\vec{\xi}_n]_b(s_2) = \frac{(b(t_j) - b(s_2))(b(s_1) - b(t_{j-1}))}{b(t_j) - b(t_{j-1})} + [\vec{\xi}_n]_b(s_1)[\vec{\xi}_n]_b(s_2), \]

which completes the proof. \qed

For $j = 1, \ldots, n$ let $\alpha_j = (1/\int_{t_{j-1}}^{t_j} (h(s))^2 ds)^{1/2} \chi_{(t_{j-1}, t_j)} h$. Let $V$ be the subspace of $L^2[0, t]$ generated by $\{\alpha_1, \ldots, \alpha_n\}$ and let $V^\perp$ be the orthogonal complement of $V$. Let $P \colon L^2[0, t] \to V$ and $P^\perp \colon L^2[0, t] \to V^\perp$ be the orthogonal projections. Let $\mathcal{M}(L^2[0, t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L^2[0, t]$ and let $S_{w_\varphi}$ be the space of all functions $F$ which for $\sigma \in \mathcal{M}(L^2[0, t])$ have the form

\[ F(x) = \int_{L^2[0, t]} \exp\{i(v, x)\} d\sigma(v) \quad (2) \]

for $w_\varphi$-a.e. $x \in C[0, t]$. Note that $S_{w_\varphi}$ is a Banach algebra [4].

By the definition of the Paley-Wiener-Zygmund integral, we have the following lemma.

**Lemma 3.5** Let $v \in L^2[0, t]$. Then for $w_\varphi$-a.e. $x \in C_0[0, t]$

\[ (v, [Z(x, \cdot)]) = (P(vh), x). \]
Theorem 3.6 Let $F \in S_{\varphi}$ and $\sigma \in \mathcal{M}(L_2[0,t])$ be related by (2). Then $G(Z(x, \cdot))$ is (Wiener) $w_\varphi$-integrable. Moreover, $E[F(Z(x, \cdot))|Z_n](\tilde{\xi}_n)$ exists for a.e. $\tilde{\xi}_n \in \mathbb{R}^n$ (hence for $P_{Z_n}$-a.e. $\tilde{\xi}_n \in \mathbb{R}^n$) and it is given by

$$E[F(Z(x, \cdot))|Z_n](\tilde{\xi}_n) = \int_{L_2[0,t]} \exp\{i(v, [\tilde{\xi}_n]_b)\} \exp\left\{-\frac{\|P^\perp(vh)\|^2}{2}\right\} \sigma(v).$$

Proof. Since $F$ is bounded, it is $w_\varphi$-integrable. For a.e. $\tilde{\xi}_n \in \mathbb{R}^n$ (hence $P_{Z_n}$-a.e. $\tilde{\xi}_n \in \mathbb{R}^n$), we have by Theorems 1.1, 2.12 and Lemma 3.5

$$E[G(Z(x, \cdot))|Z_n](\tilde{\xi}_n) = \int_{L_2[0,t]} \int_{C[0,t]} \exp\{i(v, Z(x, \cdot) - [Z(x, \cdot)] + [\tilde{\xi}_n]_b)\} dw_\varphi(x) d\sigma(v)$$

$$= \int_{L_2[0,t]} \exp\{i(v, [\tilde{\xi}_n]_b)\} \int_{C[0,t]} \exp\{i(vh - P(vh), x)\} dw_\varphi(x) d\sigma(v)$$

$$= \int_{L_2[0,t]} \exp\{i(v, [\tilde{\xi}_n]_b)\} \int_{C[0,t]} \exp\{i(P^\perp(vh), x)\} dw_\varphi(x) d\sigma(v)$$

$$= \int_{L_2[0,t]} \exp\{i(v, [\tilde{\xi}_n]_b)\} \exp\left\{-\frac{\|P^\perp(vh)\|^2}{2}\right\} \sigma(v)$$

where the last equality follows from the following integral formula

$$\int_{\mathbb{R}} \exp\{-au^2 +ibu\} du = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\}$$

for $a \in \mathbb{C}_+$ and any real $b$. Now, the proof is completed. \qed

ACKNOWLEDGEMENTS. This work was supported by Kyonggi University Research Grant 2012.

References


Received: March 14, 2013