Morera’s Theorem for Functions of a Hyperbolic Variable

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Abstract

In this paper I consider functions defined on the plane of hyperbolic numbers satisfying natural differential equations of the first order. It is well known that these functions are the analogue in the hyperbolic plane of holomorphic functions in the complex plane. I show that they satisfy a Morera-like theorem.

1. Introduction

The hyperbolic numbers are a two-dimensional commutative algebra over the real numbers defined by \( \mathbb{D} = \{ z = x + ky; k^2 = 1; x, y \in \mathbb{R}; k \notin \mathbb{R} \} \). Given \( z_1 = x + ky \) and \( z_2 = r + ks \), it follows naturally that addition and multiplication in \( \mathbb{D} \) are defined by

\[
    z_1 + z_2 := (x + ky) + (r + ks) = (x + r) + k(y + s),
\]

and

\[
    z_1 \cdot z_2 := (x + ky)(r + ks) = (xr + ys) + k(xs + yr).
\]

The hyperbolic conjugate is defined by \( z^* := x - ky \), from which we immediately notice that

\[
    z \cdot z^* = x^2 - y^2 \in \mathbb{R}.
\]
Therefore, at least when $x^2 - y^2 \neq 0$, a hyperbolic number $z$ admits an inverse given by

$$z^{-1} = \frac{z^*}{x^2 - y^2}.$$  

An immediate consequence of this consideration is the fact that $\mathbb{D}$ admits zero divisors since if $x$ and $y$ are non-zero real numbers and $x^2 - y^2 = 0$, the product $z \cdot z^*$ vanishes, even though both $z$ and $z^*$ are different from zero. More precisely, all zero divisors are of the form $z = x + ky$, with $y = \pm x$. For a more in depth analysis of hyperbolic numbers, I refer the reader to [4].

2. Holomorphicity in the Hyperbolic Setting

Recall that a complex holomorphic function is a function $g : \Omega \subset \mathbb{C} \to \mathbb{C}$ that is complex differentiable at every point in its domain $\Omega$, [1].

To extend this notion of holomorphicty to a function $f : \mathbb{D} \to \mathbb{D}$, we must define the derivative of a hyperbolic-valued function of a hyperbolic variable $z$ as the following limit:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}, \quad (1)$$

where $\Delta z$ is taken never to be a zero-divisor. If this limit exists for all points in a domain, we say that $f$ is hyperbolic holomorphic on the domain itself. Notice that the limit that defines the derivative of a function on $\mathbb{D}$ is the same regardless of the path along which $\Delta z$ approaches zero. In particular, the limit can be calculated by using those paths for which $\Delta x = 0$ or $\Delta y = 0$ (notice in particular that such paths never intersect the set of zero divisors). In the case in which $\Delta y = 0$, formula (1) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial f}{\partial x}, \quad (2)$$

and in the case in which $\Delta x = 0$ we obtain

$$f'(z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{k\Delta y} = k \frac{\partial f}{\partial y}. \quad (3)$$

The equality of the derivative of $f$ taken from both paths therefore yields

$$\frac{\partial f}{\partial x} = k \frac{\partial f}{\partial y}. \quad (4)$$
In particular, we have that the derivative of a hyperbolic holomorphic function can be given by the formula (analogous to the one that is known for the complex case):

\[ f'(z) := \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = k \frac{\partial f}{\partial y}. \]

Consequently, if we rewrite the function \( f \) in terms of its hyperbolic components, i.e. \( f(z) = f(x, y) = U(x, y) + kV(x, y) \), with \( U, V \) real valued functions of two real variables, we may also say that \( f \) is hyperbolic holomorphic if the following version of the Cauchy-Riemann system is satisfied:

\[ U_x = V_y \] (5)
\[ U_y = V_x. \] (6)

The reader is again referred to [4] for a more detailed discussion of this topic.

We conclude this introductory section by noting that the Cauchy Integral Theorem in the complex case, [1], states that if \( f(z), z \in \mathbb{C} \), is analytic in some simply connected region \( \Omega \), then

\[ \oint_{\gamma} f(z) \, dz = 0 \] (7)

for any closed contour \( \gamma \) completely contained in \( \Omega \). As shown in [2], the same results holds for hyperbolic holomorphic functions. Specifically, Catoni and his coauthors have proved the following result:

**Theorem 2.1.** Hyperbolic Cauchy Theorem. Let us consider a function \( f(z) \) of the hyperbolic variable, differentiable in a simply connected domain \( \Omega \). Then the value of the integral on a closed curve \( \gamma \) lying inside \( \Omega \) is zero.

3. Morera’s Theorem

In this section we will show that it is possible to prove also a hyperbolic version of the classical Morera’s theorem. We first need an auxiliary result.

**Lemma 3.1.** Let \( f(x, y) = U(x, y) + kV(x, y) \) be a hyperbolic-holomorphic function in some domain of \( \mathbb{D} \). Then the derivative \( \frac{\partial f}{\partial z} \) of \( f \) is also hyperbolic-holomorphic.

**Proof.** By taking the derivative of \( f \) with respect to \( x \), we obtain that the derivative of \( f \) is given by \( \frac{\partial f}{\partial x}(x, y) = U_x(x, y) + kV_x(x, y) \). To prove our lemma
it is enough to show that such derivative satisfies the hyperbolic Cauchy-Riemann system, that is,
\[
(U_x)_x = (V_x)_y
\]
\[
(U_y)_x = (V_x)_y.
\]
By substitution, equation (8) becomes \((V_y)_x = (V_x)_y\) or \(V_{yx} = V_{xy}\), which is true by Clairaut’s Theorem, and equation (9) becomes \((U_x)_y = (U_y)_x\) or \(U_{xy} = U_{yx}\), true for the same reason. Thus, derivatives of hyperbolic-holomorphic functions are also hyperbolic-holomorphic.

The previous result implies that the converse of the Cauchy Integral Theorem is also true, i.e. that a Morera-like theorem for the hyperbolic valued functions is possible.

**Theorem 3.2.** Hyperbolic Morera’s Theorem. A continuous, hyperbolic-valued function \(f(z)\) defined on a connected open set \(\Omega\) in the hyperbolic plane that satisfies \(\oint_\gamma f(z)dz = 0\) for every closed piecewise \(C^1\) curve \(\gamma\) in \(\Omega\) must be holomorphic on \(\Omega\).

**Proof.** In order to show that \(f\) is hyperbolic analytic, we will show that there exists a hyperbolic analytic function \(F\) such that \(f\) is its derivative. The result then follows from the previous lemma. To clarify the proof, we first consider a very special (but illustrative case), by assuming that \(\Omega\) is a disk centered at the point \((x_0, y_0)\). Let now \(z \in \Omega\), \(z = x + ky\), and define
\[
F(x + ky) := \int_{y_0}^{y} kf(x_0 + kt)dt + \int_{x_0}^{x} f(t + ky)dt.
\]
We notice, then, that \(\frac{\partial F}{\partial x} = f(x + ky)\).

From our hypothesis on \(\Omega\), we can also write (10) as
\[
F(x + ky) = \int_{x_0}^{x} f(t + ky_0)dt + \int_{y_0}^{y} kf(x + kt)dt
\]
from which we find that \(\frac{\partial F}{\partial y} = kf(x + ky)\). Therefore, \(f = F'\), and because \(F\) is hyperbolic holomorphic, \(F'\) is also hyperbolic holomorphic. In the general case of a connected open set \(\Omega\), we cannot guarantee the existence of a point \((x_0, y_0)\) connected to any other point in \(\Omega\) through a rectangle fully included in \(\Omega\) itself. To remedy this problem, we could construct a sequence of rectangles connecting \((x_0, y_0)\) to \((x, y)\), or more simply take any continuous arc \(\gamma\) connecting \((x_0, y_0)\) to \((x, y)\). By the hypothesis on the function \(f\), one would have that the function
\[
F(z) = \int_{\gamma} f(t)dt
\]
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is well defined and does not depend on the choice of $\gamma$. Now, by the continuity of $f$, and the definition of the derivative, one can see that $F(z)$ is in fact differentiable at any point and its derivative is $f(z)$. This is, once again, sufficient to conclude the proof.

The fact that the Morera’s Theorem holds for functions on the hyperbolic plane allows us to obtain a few important corollaries.

**Corollary 3.3.** Hyperbolic Painlevé’s Theorem If open plane domains $D_1$ and $D_2$ abut along the rectifiable arc $C$, if $f_j(z)$ is hyperbolic holomorphic on $D_j$ and continuous on $D_j \cup C$, $j = 1, 2$, and if $f_1(z) = f_2(z)$ on $C$, then $f_1(z)$ and $f_2(z)$ are hyperbolic holomorphic continuations of one another across $C$.

**Proof.** Let $D = D_1 \cup D_2$. Consider an arbitrary piecewise smooth closed contour $\gamma$ so that its interior lies in $D$. If it lies entirely in $D_1$ or $D_2$, then, by the Cauchy Integral Formula, $\oint_{\gamma} f(z)\,dz = 0$. If $\gamma$ is contained partly in $D_1$ and partly in $D_2$, denoted by $\gamma_1$ and $\gamma_2$, respectively, and the arc $C \cap \gamma$ is denoted by $c$, then again by the Cauchy Integral Formula, $\oint_{\gamma_1 \cup c} f(z)\,dz = 0$ and $\oint_{\gamma_2 \cup c} f(z)\,dz = 0$. Hence, $\oint_{\gamma} f(z)\,dz = 0$. Thus, by the previous theorem, $f$ is hyperbolic analytic.

**Remark 3.4** Because of the special form of the Cauchy Integral found in $\mathbb{D}$, we cannot expect to be able to prove any analogue of the Mittag-Leffler Theorem. This is consistent with the fact that $\mathbb{D}$ has a hyperbolic, rather than elliptic, system.

**Corollary 3.5.** Let $\{f_n\}$ be a sequence of hyperbolic holomorphic functions on a domain $\Omega$. Assume that the sequence $\{f_n\}$ is uniformly convergent on compact subsets of $\Omega$ to a function $f$. Then $f$ is also hyperbolic holomorphic.

**Proof.** It is enough to show that $\oint_{\gamma} f(t)\,dt = 0$ for any closed curve $\gamma$ contained in $\Omega$. But since $f$ is the uniform limit of the sequence $\{f_n\}$, since the integrals of these functions vanish on $\gamma$ and since $\gamma$ is a compact set, this follows immediately, and proves the holomorphicity of $f$.

**References**


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