Some Identities on the $q$-Genocchi Polynomials with Weak Weight $\alpha$ and Bernstein Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract
In this paper, by using fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give some interesting relationship between Bernstein polynomials and $q$-Genocchi polynomials with weak weight $\alpha$.

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1 Introduction
Let $p$ be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value is defined by $|x|_p = \frac{1}{p^r}$, where $x = p^r s t (r \in \mathbb{Q} \text{ and } s, t \in \mathbb{Z}$ with $(s, t) = (p, s) = (p, t) = 1)$. In this paper we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$ as an indeterminate. The $q$-number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q},$$

see [1, 2, 3, 4, 5, 6, 7]. Note that $\lim_{q \to 1} [x]_q = x$. For

$$f \in UD(\mathbb{Z}_p) = \{f | f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$
the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1 + q^p N} \sum_{x=0}^{p^{N-1}} f(x)(-q)^x, \text{ see } [2, 3]. \quad (1.1)$$

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p \leq 1$, the $q$-Genocchi polynomials $\widetilde{G}_{n,q}^{(\alpha)}$ with weak weight $\alpha$ are defined by

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = n \int_{\mathbb{Z}_p} (x + y)^{n-1} d\mu_{-q}(y). \quad (1.2)$$

In the special case, $x = 0$, $\widetilde{G}_{n,q}^{(\alpha)}(0) = \widetilde{G}_{n,q}^{(\alpha)}$ are called the $n$-th $q$-Genocchi numbers with weak weight $\alpha$ (see [1]). In this paper we investigate some relations between Bernstein polynomials and $q$-Genocchi numbers with weak weight $\alpha$. From these relations, we derive some interesting identities on the $q$-Genocchi numbers with weak weight $\alpha$.

## 2 $q$-Genocchi polynomials with weak weight $\alpha$ and Bernstein polynomials

The following elementary properties of $q$-Genocchi polynomials $\widetilde{G}_{n,q}^{(\alpha)}(x)$ are readily derived from (1.1) and (1.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [1], [7].

**Proposition 2.1** ([7]) For any positive integer $n$ and $\alpha \in \mathbb{Z}$, we have

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \widetilde{G}_{k,q}^{(\alpha)} = \left(x + \widetilde{G}_q^{(\alpha)}\right)^n.$$ 

By (2.5), we have

$$\widetilde{G}_{q^{-1}}^{(\alpha)}(1 - t, -x) = \sum_{n=0}^{\infty} (-1)^{n-1} \widetilde{G}_{n,q^{-1}}^{(\alpha)}(1 - x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \widetilde{G}_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}.$$ 

Thus we have the following theorem.

**Theorem 2.2** For any positive integer $n$, we have

$$\widetilde{G}_{n,q}^{(\alpha)}(x) = (-1)^{n-1} \widetilde{G}_{n,q^{-1}}^{(\alpha)}(1 - x)$$

For $q$-Genocchi numbers with weak weight $\alpha$, we have the following theorem.
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**Proposition 2.3** ([7]) For $n \in \mathbb{Z}_+$, we have
\[
\tilde{G}^{(\alpha)}_{n,q} = 0, \quad q^\alpha \tilde{G}^{(\alpha)}_{n,q}(1) + \tilde{G}^{(\alpha)}_{n,q} = \begin{cases} 
[2]_{q^n}, & \text{if } n = 1, \\
0, & \text{if } n > 1.
\end{cases}
\]
with the usual convention about replacing $(\tilde{G}^{(\alpha)}_q)^n$ by $\tilde{G}^{(\alpha)}_{n,q}$ in the binomial expansion.

**Proposition 2.4** ([7]) For $n \in \mathbb{Z}_+$, we have
\[
\tilde{G}^{(\alpha)}_{0,q} = 0, \quad q^\alpha (\tilde{G}^{(\alpha)}_q + 1)^n + \tilde{G}^{(\alpha)}_{n,q} = \begin{cases} 
[2]_{q^n}, & \text{if } n = 1, \\
0, & \text{if } n > 1,
\end{cases}
\]
with the usual convention of replacing $(\tilde{G}^{(\alpha)}_q)^n$ by $\tilde{G}^{(\alpha)}_{n,q}$.

By Proposition 2.1 and Proposition 2.3, we obtain
\[
\tilde{G}^{(\alpha)}_{n,q}(2) = \sum_{l=0}^{n} \binom{n}{l} 2^{n-l} \tilde{G}^{(\alpha)}_{l,q} = \left(\tilde{G}^{(\alpha)}_q + 1 + 1\right)^n
\]
\[
= \frac{n[2]_{q^n}}{q^\alpha} - \frac{1}{q^\alpha} \sum_{l=1}^{n} \binom{n}{l} \tilde{G}^{(\alpha)}_{l,q}(1)
\]
\[
= \frac{n[2]_{q^n}}{q^\alpha} + \frac{1}{q^{2\alpha}} \tilde{G}^{(\alpha)}_{n,q} \text{ if } n > 1.
\]

Therefore, we obtain the following theorem.

**Theorem 2.5** For $n \in \mathbb{N}$ with $n > 1$, we have
\[
q^{2\alpha} \tilde{G}^{(\alpha)}_{n,q}(2) = n[2]_{q^n} q^\alpha + \tilde{G}^{(\alpha)}_{n,q}.
\]

By Theorem 2.5, we have the following corollary.

**Corollary 2.6** For $n \in \mathbb{N}$ with $n > 1$, we have
\[
\tilde{G}^{(\alpha)}_{n,q-1}(2) = n[2]_{q^n} + q^{2\alpha} \tilde{G}^{(\alpha)}_{n,q-1}.
\]

As well known definition, Bernstein polynomials of degree $n$ are given by
\[
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where } x \in [0,1], n, k \in \mathbb{Z}_+, \text{ see [5, 6].} \quad (2.1)
\]

By (2.1), we get the symmetry of Bernstein polynomials as follows:
\[
B_{k,n}(x) = B_{n-k,n}(1-x).
\]
By Theorem 2.2 and Corollary 2.6, we have
\[
\int_{\mathbb{Z}_p} (1 - x)^n d\mu_{-q^a}(x) = (-1)^n \int_{\mathbb{Z}_p} (x - 1)^n d\mu_{-q^a}(x)
\]
\[
= (-1)^n \frac{\tilde{G}_{n+1,q}^{(\alpha)}(-1)}{n+1}
\]
\[
= (-1)^n \frac{(-1)^n \tilde{G}_{n+1,q}^{(\alpha)}(2)}{n+1}
\]
\[
= [2]_{q^a} + q^{2\alpha} \frac{\tilde{G}_{n+1,q}^{(\alpha)}}{n+1}.
\]
Let us take the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree $n$ as follows:
\[
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q^a}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} x^k (1 - x)^{n-k} d\mu_{-q^a}(x)
\]
\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} (1 - x)^{n-l} d\mu_{-q^a}(x)
\]
\[
= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \left( [2]_{q^a} + q^{2\alpha} \frac{\tilde{G}_{n-l+1,q}^{(\alpha)}}{n-l+1} \right).
\]
Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.7** Let $n, k \in \mathbb{Z}_+$ with $n > k + 1$. Then we have
\[
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q^a}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_{q^a} + q^{2\alpha} \frac{\tilde{G}_{n-k+1,q}^{(\alpha)}}{n-k+1} \right).
\]
Moreover,
\[
\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{\tilde{G}_{l+k+1,q}^{(\alpha)}}{l+k+1} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left( [2]_{q^a} + q^{2\alpha} \frac{\tilde{G}_{n-k+1,q}^{(\alpha)}}{n-k+1} \right).
\]
Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we get
\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^a}(x)
\]
\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^a} + q^{2\alpha} \frac{\tilde{G}_{n_1+n_2-l+1,q}^{(\alpha)}}{n_1+n_2-l+1} \right). \quad (2.4)
\]
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Therefore, by (2.4), we have the following theorem.

**Theorem 2.8** For $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$, we have

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x)$$

$$= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^\alpha} + q^{2\alpha} \frac{\tilde{G}_{n_1+n_2-l+1,q^{-1},l}}{n_1+n_2-l+1} \right).$$

From the binomial theorem, we can derive the following equation.

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-q^\alpha}(x)$$

$$= \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{n_1+n_2-2k} (-1)^{l} \binom{n_1+n_2-2k}{l} \frac{\tilde{G}_{2k+l+1,q}}{2k+l+1}. \quad (2.5)$$

Thus, by (2.5) and Theorem 2.8, we obtain the following corollary.

**Corollary 2.9** Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have

$$\sum_{l=0}^{n_1+n_2-2k} (-1)^{l} \binom{n_1+n_2-2k}{l} \frac{\tilde{G}_{2k+l+1,q}}{2k+l+1}$$

$$= \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left( [2]_{q^\alpha} + q^{2\alpha} \frac{\tilde{G}_{n_1+n_2-l+1,q^{-1},l}}{n_1+n_2-l+1} \right).$$

For $x \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk + 1$. Then we take the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ for the Bernstein polynomials of degree $n$ as follows:

$$\int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}(x) \cdots B_{k,n_s}(x)}_{s\text{-times}} d\mu_{-q^\alpha}(x)$$

$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \int_{\mathbb{Z}_p} x^{sk}(1-x)^{n_1+\cdots+n_s-sk} d\mu_{-q^\alpha}(x)$$

$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left( [2]_{q^\alpha} + q^{2\alpha} \frac{\tilde{G}_{n_1+\cdots+n_s-l+1,q^{-1},l}}{n_1+\cdots+n_s-l+1} \right).$$

Therefore we have the following theorem.
Theorem 2.10 For \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + \cdots + n_s > sk + 1 \). Then we get

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) d\mu_{-q^n}(x) = \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left[ 2q^n + q^{2\alpha} \frac{\widetilde{G}_{n_1+\cdots+n_s-l+1,q^{-1}}}{n_1 + \cdots + n_s - l + 1} \right].
\]

By the definition of Bernstein polynomials and the binomial theorem, we easily get

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) d\mu_{-q^n}(x) = \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left( \frac{\widetilde{G}_{sk+l+1,q}}{sk + l + 1} \right). \tag{2.6}
\]

Therefore, by Theorem 2.10 and (2.6), we have the following corollary.

Corollary 2.11 For \( s \in \mathbb{N} \) with \( s \geq 2 \), let \( n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+ \) with \( n_1 + \cdots + n_s > sk + 1 \). Then we have

\[
\sum_{l=0}^{n_1+\cdots+n_s-sk} (-1)^{l} \binom{n_1 + \cdots + n_s - sk}{l} \frac{\widetilde{G}_{sk+l+1,q}}{sk + l + 1} \]

\[
= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \left[ 2q^n + q^{2\alpha} \frac{\widetilde{G}_{n_1+\cdots+n_s-l+1,q^{-1}}}{n_1 + \cdots + n_s - l + 1} \right].
\]

References


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