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New Interval Oscillation Criteria for Forced Second-Order Differential Equations with Nonlinear Damping

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Abstract

We introduce and prove some new interval oscillation criteria for a general class of forced second-order differential equations

$$(E): \quad \big(r(t)k_1(x,x')\big)' + p(t)k_2(x,x')x' + q(t)f(x) = e(t), \ t \geq t_0,$$

where the functions $k_1(u,v)$, $k_2(u,v)$ and $f(u)$ satisfy some general conditions. Our interval oscillation criteria and their proofs are different than previously published ones, and it is based on (i): a construction of a global supersolution of the generalized Riccati differential equation (R): $\omega' = A_1(t)|\omega(t)|^\beta + A_2(t)|\omega(t)|^\delta + B(t)$, $t \geq T$, by using the classic Riccati transformation of a nonoscillatory solution of the main equation (E), on (ii): a construction of a pair of local subsolutions of equation (R) under a new oscillatory condition on the coefficients of the main equation (E) and on (iii): a pointwise comparison principle between all sub- and supersolutions of equation (R).

Mathematics Subject Classification: 34C10

Keywords: interval criteria, oscillation, second order, forced term, nonlinear damping, Riccati differential equation, pointwise comparison principle, sub and supersolutions, blow-up argument

1 Introduction and main results

In the paper, we study the oscillation of the following general class of forced second-order differential equations with nonlinear damping:

$$\big(r(t)k_1(x,x')\big)' + p(t)k_2(x,x')x' + q(t)f(x) = e(t),\ t \ge t_0 > 0, \tag{1}$$

where the coefficients $r(t)$, $p(t)$, and $q(t)$, as well as the functions $k_1(u,v)$ and $k_2(u,v)$ are continuous in all their variables, $k_1 \in C^1(\mathbb{R}^2,\mathbb{R})$ and solution $x = x(t)$, $x \in C^2((t_0,\infty),\mathbb{R}) \cap C([t_0,\infty),\mathbb{R})$. A function $x(t)$ is said to be oscillatory if there is a sequence $t_n \ge t_0$ such that $x(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$.

An oscillation criterion of interval type means that we do not require some informations about the coefficients $r(t)$, $p(t)$, and $q(t)$ on the whole half-line $[t_0,\infty)$ than only on each given pair of intervals. Precisely, we allways suppose that for any $T \ge t_0$ there are a_1, b_1, a_2, b_2, $T \le a_1 < b_1 \le a_2 < b_2$ such that:

$$e(t) \le 0 \text{ on } [a_1,b_1] \quad \text{and} \quad e(t) \ge 0 \text{ on } [a_2,b_2], \tag{2}$$

$$r(t) > 0 \quad \text{on } [a_1,b_1] \cup [a_2,b_2]. \tag{3}$$

In the case when $f(u)$ is not necessarily a monotone function, we suppose that $f(u)$ satisfies:

$$f(u)/u \ge K|u|^{\gamma-1} \ \text{ for some } K > 0 \text{ and } \gamma \ge 1 \text{ and all } u \in \mathbb{R},\ u \ne 0. \tag{4}$$

For the functions $k_1(u,v)$ and $k_2(u,v)$, which are appearing respectively in the second order differential operator $(r(t)k_1(x,x'))'$ and the nonlinear damped term $p(t)k_2(x,x')x'$, we impose the following two assumptions that hold for all $(u,v) \in \mathbb{R}^2$, $u \ne 0$:

$$k_1(u,v)v \ge \alpha_1 |k_1(u,v)|^{\beta} |u|^{2-\beta} \ \text{ for some } \alpha_1 > 0,\ \beta > 1, \tag{5}$$

$$k_2(u,v)\operatorname{sgn}(u)|u|^{\delta-1}v \ge \alpha_2 |k_1(u,v)|^{\delta} \ \text{ for some } \alpha_2 > 0,\ \delta > 1. \tag{6}$$

In particular for $\beta = \delta = 2$, conditions (5) and (6) are equivalent with the following basic ones:

(i) $k_1(u,v)v \ge \alpha_1 k_1^2(u,v)$ for some $\alpha_1 > 0$ and all $(u,v) \in \mathbb{R}^2$;
(ii) $k_2(u,v)uv \ge \alpha_2 k_1^2(u,v)$ for some $\alpha_2 \ge 0$ and all $(u,v) \in \mathbb{R}^2$.

The assumptions (i)-(ii) were considered for the first time in [10] (see also [15] and [18]) in the study of oscillation of equation (1) with $e(t) \equiv 0$ (when $e(t) \not\equiv 0$ see for instance [12]).

In particular for $\beta = \delta = (\alpha+1)/\alpha$, conditions (5) and (6) generalize in some sense the following assumptions:

(iii) $k_1(u,v)v \geq \alpha_1 |k_1(u,v)|^{\frac{\alpha+1}{\alpha}} |u|^{\frac{\alpha-1}{\alpha}}$ for some $\alpha_1 > 0$ and $\alpha \geq 1$ and all $(u,v) \in \mathbb{R} \times \mathbb{R}$, $v \neq 0$;

(iv) $k_2(u,v)|u|^{\frac{1}{\alpha}} v \geq \alpha_2 |k_1(u,v)|^{\frac{\alpha+1}{\alpha}}$ for some $\alpha_2 > 0$ and all $(u,v) \in \mathbb{R}^2$.

Under assumptions (2)-(4) and (iii)-(iv), authors in [7] generalize known interval oscillation criteria obtained in [1] and [22] from linear forced second-order differential equations to general type of equations such as (1), see for instance [7, Theorem 3.2]. Furthermore, under the assumptions (iii)-(iv) but assuming Kamenev type oscillation criterion, the oscillation of equation (1) has been studied in [2], see also [17, 5, 16, 3, 14]. When $e(t) \equiv 0$, the oscillation of equation (1) with assumptions (i)-(ii) has been widely considered, see for instance [10, 15, 18, 11, 6, 9, 21, 4].

According to a discussion from [11], one can take $k_1(u,v) = v$ so that the following very particular case of assumption (5) for $\beta = 2$ holds: $k_1(u,v)v = k_1^2(u,v)$. Putting this equality into assumption (6), we conclude that $k_2(u,v)$ can be discontinuous at $u = 0$ which is a contradiction with the assumption that $k_2(u,v)$ is a continuous function in all its variable. In such a case, instead of assumption (6) we propose the more general one:

$$k_2(u,v)uv \geq 0 \quad \text{for all } (u,v) \in \mathbb{R}^2, \tag{1.6$_w$}$$

which unlike (6) does not cause any problem with (5). Hence, in the second case of Theorem 1.1, instead of the pair of assumptions (5) and (6), we propose to consider (5) and $(1.6)_w$.

It is not difficult to see, for more details see Section 2, that if $\beta > 1$, then assumption (5) can be very restrictive for the linear and half-linear case, that is, for $k_1(u,v) = \Phi(u)v$ and $k_1(u,v) = \Phi(u)|v|^{\sigma-2}v$, $\sigma \geq 1$. In order to include in our study here these two important cases for all $\beta > 1$, instead of assumption (5) we propose the next one:

$$k_1(u,v)v \geq 0 \quad \text{for all } (u,v) \in \mathbb{R}^2. \tag{1.5$_w$}$$

Hence, in the third case of Theorem 1.1 below, instead of the pair of assumptions (5) and (6), we consider $(1.5)_w$ and (6).

Besides assumptions (5), (6), $(1.5)_w$, and $(1.6)_w$, we additionaly suppose some new interval oscillation criteria which are different than previously published ones. In these criteria, besides the coefficients $r(t)$ and $p(t)$, a function $Q(t)$ also plays an important role, which is defined by:

$$Q(t) = \begin{cases} Kq(t) & \text{if } \gamma = 1, \\ \gamma(\gamma-1)^{\frac{1-\gamma}{\gamma}} [Kq(t)]^{\frac{1}{\gamma}} |e(t)|^{\frac{\gamma-1}{\gamma}} & \text{if } \gamma > 1. \end{cases} \tag{7}$$

Theorem 1.1. (without monotonicity of $f(u)$) *Let assumption* (4) *hold and* $Q(t)$ *be given by* (7). *Let for any large enough* $T \geq t_0$ *there exist* a_1, b_1, a_2, b_2, $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that* (2) *and* (3) *be satisfied. Let* $p(t) \geq 0$ *and* $q(t) \geq 0$ *on* $[a_1, b_1] \cup [a_2, b_2]$. *For a function* $C(t)$, *let* $c_i := \int_{a_i}^{b_i} C(\tau) d\tau > 0$, $i \in \{1, 2\}$. *Then equation* (1) *is oscillatory provided one of the following hypotheses is satisfied:*

(H_1) *conditions* (5) *and* (6) *hold and there are a real parameter* $\lambda > 0$ *and a function* $C = C(t)$, $C \in L^1((a_1, b_1) \cup (a_2, b_2), \mathbb{R}_+)$, *such that*

$$\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin\frac{\pi}{\beta} \min\left\{\frac{\alpha_1}{(\lambda r(t))^{\beta-1}}, \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)}, \lambda Q(t)\right\}, \quad t \in [a_i, b_i],\ i \in \{1, 2\}; \tag{8}$$

(H_2) *conditions* (5) *and* $(1.6)_w$ *hold and there are a real parameter* $\lambda > 0$ *and a function* $C = C(t)$, $C \in L^1((a_1, b_1) \cup (a_2, b_2), \mathbb{R}_+)$, *such that*

$$\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin\frac{\pi}{\beta} \min\left\{\frac{\alpha_1}{(\lambda r(t))^{\beta-1}}, \lambda Q(t)\right\}, \quad t \in [a_i, b_i],\ i \in \{1, 2\}; \tag{9}$$

(H_3) *conditions* $(1.5)_w$ *and* (6) *hold and there are a real parameter* $\lambda > 0$ *and a function* $C = C(t)$, $C \in L^1((a_1, b_1) \cup (a_2, b_2), \mathbb{R}_+)$, *such that*

$$\frac{1}{c_i} C(t) \leq \frac{\delta}{2\pi} \sin\frac{\pi}{\delta} \min\left\{\frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)}, \lambda Q(t)\right\}, \quad t \in [a_i, b_i],\ i \in \{1, 2\}. \tag{10}$$

In the next section, we will give some important examples which illustrate our main results.

Remark 1.2. Although coefficients $p(t)$ and $q(t)$ are supposed to be nonnegative functions on $[a_1, b_1] \cup [a_2, b_2]$, they can change sign on $[t_0, \infty)$. For instance, if $e(t) = \cos(t)$, then functions $p(t) = q(t) = \sin(t)$ change sign on $[t_0, \infty)$ but they are nonnegative on $[a_1, b_1] \cup [a_2, b_2]$, where $a_1 = (\frac{1}{2} + 2n)\pi$, $b_1 = (1 + 2n)\pi$, $a_2 = (2 + 2n)\pi$, and $b_2 = (\frac{5}{2} + 2n)\pi$, $n \in \mathbb{N}$. At the same time $e(t) \leq 0$ on $[a_1, b_1]$ and $e(t) \geq 0$ on $[a_2, b_2]$. □

In the case when $f(u)$ is a monotone function, then instead of assumptions (4), (5) and (6), we suppose that $f \in C^1(\mathbb{R})$ and for some $\beta > 1$, $K > 0$, and $\alpha_1 > 0$.

$$u f(u) > 0 \quad \text{and} \quad |f(u)|^{\beta-2} f'(u) \geq K \quad \text{for all } u \neq 0, \tag{11}$$

$$k_1(u, v) v \geq \alpha_1 |k_1(u, v)|^{\beta} \quad \text{for all } (u, v) \in \mathbb{R}^2,\ u \neq 0, \tag{12}$$

and for some $\delta > 1$ and $\alpha_2 > 0$,

$$k_2(u, v) \mathrm{sgn}(u) |f(u)|^{\delta-1} v \geq \alpha_2 |k_1(u, v)|^{\delta} \quad \text{for all } (u, v) \in \mathbb{R}^2. \tag{13}$$

Using the same technique as in Theorem 1.1 we prove the second main result of the paper.

Theorem 1.3. *Let assumptions* (11)*,* (12) *and* (13) *hold. Let for any large enough* $T \geq t_0$ *there exist* a_1, b_1, a_2, b_2*,* $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that* (2) *and* (3) *be satisfied. Let* $p(t) \geq 0$ *and* $q(t) \geq 0$ *on* $[a_1, b_1] \cup [a_2, b_2]$*. If there are a real parameter* $\lambda > 0$ *and a function* $C = C(t)$*,* $C \in L^1((a_1, b_1) \cup (a_2, b_2), \mathbb{R}_+)$*, such that* $c_i := \int_{a_i}^{b_i} C(\tau) d\tau > 0$*,* $i \in \{1, 2\}$ *and*

$$\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_1 K}{(\lambda r(t))^{\beta-1}}, \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)}, \lambda q(t) \right\}, \quad t \in [a_i, b_i], \tag{14}$$

then equation (1) *is oscillatory.*

Remark 1.4. In recently published author's paper [9], the oscillation of equation (1) has been considered especially for $e(t) \equiv 0$ and with assumption (4) only in the particular case $\gamma = 2$. Also, in (5) and (6) it has been supposed that $\beta = 2m$ and $\delta = 2n$, where $m, n \in \mathbb{N}$. Here in contrast to [9], the most difficulties are appearing because $e(t) \not\equiv 0$ and especially in the case $\beta \in (1, 2)$, where we can not use the blow-up property of the classic tangens function than the generalized one. □

2 Remarks, consequences and examples

Before we present some consequences of previous theorems, we firstly discuss the main three classes of the second-order differential operators $(r(t)k_1(x, x'))'$ and the damped term $p(t)k_2(x, x')x'$, in which the functions $k_1(u, v)$ and $k_2(u, v)$ satisfy the required assumptions (5), $(1.5)_w$, (6) and $(1.6)_w$ respectively.

Remark 2.1. We consider the following special case of equation (1):

$$\big(r(t)\Phi_1(x)x'\big)' + p(t)\Phi_2(x)x'^2 + q(t)f(x) = e(t), \; t \geq t_0 > 0.$$

In particular for $k_1(u, v) = \Phi_1(u)v$ and $k_2(u, v) = \Phi_2(u)v$, it is easy to check the following assertions:

- (5) and (6) with $\beta = \delta = 2$ hold if $0 \not\equiv \Phi_1(u)$, $0 \leq \Phi_1(u) \leq \frac{1}{\alpha_1}$ and $\Phi_2(u)u \geq \alpha_2 \Phi_1^2(u)$;
- (5) with $\beta = 2$ and $(1.6)_w$ hold if $0 \not\equiv \Phi_1(u)$, $0 \leq \Phi_1(u) \leq \frac{1}{\alpha_1}$ and $\Phi_2(u)u \geq 0$;
- $(1.5)_w$ and (6) with $\delta = 2$ hold if $\Phi_1(u) \geq 0$ and $\Phi_2(u)u \geq \alpha_2 \Phi_1^2(u)$.

For instance, $\Phi_1(u) = \frac{1}{\alpha_1} \sin^2 u$ and $\Phi_2(u) = \frac{\alpha_2}{\alpha_1^2} u^3$ or $\Phi_1(u) = \frac{u^2}{\alpha_1(1+u^2)}$ and $\Phi_2(u) = \frac{\alpha_2}{\alpha_1^2} u^3$. □

Remark 2.2. We consider the next particular case of equation (1), where the second order differential operator $(r(t)k_1(x, x'))'$ is the so-called generalized prescribed mean curvature operator, that is, for $\sigma \geq 1$,

$$\left(r(t)\Phi_1(x)\frac{x'}{(1+x'^2)^{\sigma/2}}\right)' + p(t)\Phi_2(x)\frac{x'^2}{(1+x'^2)^{\sigma/2}} + q(t)f(x) = e(t),\ t \geq t_0 > 0.$$

One can set $k_1(u,v) = \Phi_1(u)v(1+v^2)^{-\sigma/2}$ and $k_2(u,v) = \Phi_2(u)v(1+v^2)^{-\sigma/2}$, and the following assertions are fulfilled:

- (5) and (6) hold if $0 < \Phi_1(u) \leq \alpha_1^{-\frac{1}{\beta-1}}|u|^{\frac{\beta-2}{\beta-1}}$ and $\Phi_2(u)\operatorname{sgn}(u)|u|^{\delta-1} \geq \alpha_2|\Phi_1(u)|^\delta$ for all $u \neq 0$;
- (5) and $(1.6)_w$ hold if $0 < \Phi_1(u) \leq \alpha_1^{-\frac{1}{\beta-1}}|u|^{\frac{\beta-2}{\beta-1}}$ and $\Phi_2(u)u \geq 0$ for all $u \neq 0$;
- $(1.5)_w$ and (6) hold if $\Phi_1(u) \geq 0$ and $\Phi_2(u)\operatorname{sgn}(u)|u|^{\delta-1} \geq \alpha_2|\Phi_1(u)|^\delta$. □

Remark 2.3. As a particular case of equation (1), we consider the next equation, where the second order differential operator $(r(t)k_1(x,x'))'$ is the classic quasilinear operator in the second variable, that is,

$$\left(r(t)\Phi_1(x)|x'|^{\sigma-1}x'\right)' + p(t)\Phi_2(x)|x'|^{2\sigma} + q(t)f(x) = e(t),\ t \geq t_0 > 0, \quad \sigma \geq 1.$$

It is clear that this equation generalizes the appropriate one from Example 2.1 when $\sigma = 1$. One can set $k_1(u,v) = \Phi_1(u)|v|^{\sigma-1}v$ and $k_2(u,v) = \Phi_2(u)|v|^{2\sigma}$, and we conclude that required assumptions $(1.5)_w$ and (6) with $\delta = 2$ hold provided $\Phi_1(u) \geq 0$ and $\Phi_2(u)u \geq \alpha_2\Phi_1^2(u)$. □

Now we state the main consequences of Theorems 1.1 and 1.3.

Corollary 2.4. (without monotonicity of $f(u)$) *Let assumption* (4) *hold and $Q(t)$ be given in* (7)*. Let for any large enough $T \geq t_0$ there exist a_1, b_1, a_2, b_2, $T \leq a_1 < b_1 \leq a_2 < b_2$ such that* (2) *and* (3) *hold, and*

$$\exists \tau_0 > 0 \textit{ such that } \ b_i - a_i \geq \tau_0 > 0,\ i \in \{1,2\}. \tag{15}$$

Let r_0, p_0, and Q_0 be three positive constants such that:

$$r(t) \leq r_0,\ p(t) \geq p_0 \quad \textit{and} \quad Q(t) \geq Q_0 \quad \textit{on } [a_1,b_1] \cup [a_2,b_2]. \tag{16}$$

Then equation (1) *is oscillatory provided one of the following hypotheses is satisfied:*

(H_1) *conditions* (5) *and* (6) *hold and there is a real parameter $\lambda > 0$ such that*

$$\frac{1}{\tau_0} \leq \frac{\beta}{2\pi}\sin\frac{\pi}{\beta}\min\left\{\frac{\alpha_1}{(\lambda r_0)^{\beta-1}}, \frac{\alpha_2 p_0}{\lambda^{\delta-1}r_0^\delta}, \lambda Q_0\right\}; \tag{17}$$

(H_2) *conditions* (5) *and* $(1.6)_w$ *hold and there is a real parameter* $\lambda > 0$ *such that*

$$\frac{1}{\tau_0} \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \Big\{ \frac{\alpha_1}{(\lambda r_0)^{\beta-1}}, \lambda Q_0 \Big\}; \tag{18}$$

(H_3) *conditions* $(1.5)_w$ *and* (6) *hold and there is a real parameter* $\lambda > 0$ *such that*

$$\frac{1}{\tau_0} \leq \frac{\delta}{2\pi} \sin \frac{\pi}{\delta} \min \Big\{ \frac{\alpha_2 p_0}{\lambda^{\delta-1} r_0^{\delta}}, \lambda Q_0 \Big\}. \tag{19}$$

Corollary 2.5. ($f(u)$ is a monotone function) *Let assumptions* (11), (12) *and* (13) *hold. Let for any large enough* $T \geq t_0$ *there exist* a_1, b_1, a_2, b_2, $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that* (2), (3), *and* (15) *be satisfied. Let* r_0, p_0, *and* q_0 *be three positive constants such that:*

$$r(t) \leq r_0,\ p(t) \geq p_0 \quad \textit{and} \quad q(t) \geq q_0 \quad \textit{on } [a_1, b_1] \cup [a_2, b_2]. \tag{20}$$

Then equation (1) *is oscillatory provided there is a real parameter* $\lambda > 0$ *such that*

$$\frac{1}{\tau_0} \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \Big\{ \frac{\alpha_1 K}{\lambda^{\beta-1} r_0^{\beta}}, \frac{\alpha_2 p_0}{\lambda^{\delta-1} r_0^{\delta}}, \lambda q_0 \Big\}. \tag{21}$$

According to Corollary 2.4, we derive the next consequences.

Example 2.6. We consider the equation:

$$\big(k_1(x, x')\big)' + \sin(2mt) k_2(x, x') x' + q_0 \sin(2mt) f(x) = e_0 \sin(t),\ t \geq t_0 > 0, \tag{22}$$

where $m \in \mathbb{N}$ and the functions $f(u)$, $k_1(u,v)$ and $k_2(u,v)$ satisfy respectively assumptions (4), (5) and (6). Then equation (22) is oscillatory provided the real numbers q_0 and e_0 satisfy:

$$\begin{cases} q_0 \geq \frac{24m}{\sqrt{3} K \lambda_0 \beta \sin(\frac{\pi}{\beta})} & \text{if } \gamma = 1, \\ q_0^{\frac{1}{\gamma-1}} e_0 \geq \frac{\gamma-1}{\sin(\frac{\pi}{6m})} \left(\frac{12m}{\gamma \big(K\sqrt{3}/2\big)^{1/\gamma} \lambda_0 \beta \sin(\frac{\pi}{\beta})} \right)^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma > 1, \end{cases} \tag{23}$$

where the constants K and γ are from (4), the constant β is from (5), the constants α_1, α_2, and δ are from (5) and (6), and

$$\lambda_0 = \min \left\{ \Big(\frac{\alpha_1}{12m} \beta \sin \frac{\pi}{\beta} \Big)^{\frac{1}{\beta-1}}, \Big(\frac{\sqrt{3}\alpha_2}{24m} \beta \sin \frac{\pi}{\beta} \Big)^{\frac{1}{\delta-1}} \right\}. \tag{24}$$

Indeed, since $r(t)=1$, $p(t)=q(t)=\sin(2mt)$, and $e(t)=\sin t$, the required conditions (2), (3), (15) and (16) are fulfilled in particular for:

$$\begin{cases} a_1 = \frac{6m+1}{6m}\pi + 2n\pi, & b_1 = \frac{3m+1}{3m}\pi + 2n\pi, \\ a_2 = \frac{12m+1}{6m}\pi + 2n\pi, & b_2 = \frac{6m+1}{3m}\pi + 2n\pi, \; n \in \mathbb{N}, \end{cases}$$

and $\tau_0 = \frac{\pi}{6m}$, $r_0 = 1$, $p_0 = \frac{\sqrt{3}}{2}$,

$$Q_0 = \begin{cases} \frac{Kq_0\sqrt{3}}{2}, & \text{if } \gamma = 1, \\ \gamma(\gamma-1)^{\frac{1-\gamma}{\gamma}} \left(\frac{Kq_0\sqrt{3}}{2}\right)^{\frac{1}{\gamma}} \left(e_0 \sin\left(\frac{\pi}{6m}\right)\right)^{\frac{\gamma-1}{\gamma}}, & \text{if } \gamma > 1. \end{cases}$$

It is because we can easily check that $p(t) \ge \sqrt{3}/2$, $q(t) \ge q_0\sqrt{3}/2$ and $|e(t)| \ge e_0 \sin\left(\frac{\pi}{6m}\right)$ on $[a_1,b_1] \cup [a_2,b_2]$, and $e(t) \le 0$ on $[a_1,b_1]$ and $e(t) \ge 0$ on $[a_2,b_2]$. Also, from assumption (23) it follows that oscillation condition (17) is satisfied in particular for $\lambda = \lambda_0$, where λ_0 is determined in (24). Hence, the hypothesis $(H1)$ is fulfilled and Corollary 2.4 proves this example. □

As a simple consequence of previous example, we observe the following interesting conclusion.

Example 2.7. We consider the equation:

$$\big(k_1(x,x')\big)' + \sin(2t)k_2(x,x')x' + q_0 \sin(2t) f(x) = e_0 \sin(t), \; t \ge t_0 > 0, \tag{25}$$

where $f(u)$ satisfy (4) with $\gamma = 1$, and $k_1(u,v)$ and $k_2(u,v)$ satisfy (5) and (6) with $\alpha_1 = \alpha_2 = 1$ and $\beta = \delta = 2$. Then equation (25) is oscillatory provided $q_0 \ge 48/K$, where K is from (4). □

Analogously to Example 2.6, one can derive sufficient conditions on the constants q_0 and e_0 such that the next equation is oscillatory as follows.

Example 2.8. We consider the equation:

$$\big(k_1(x,x')\big)' + \cos(2mt)k_2(x,x')x' + q_0 \cos(2mt) f(x) = e_0 \cos(t), \; t \ge t_0 > 0, \tag{26}$$

where $m \in \mathbb{N}$ and the functions $f(u)$, $k_1(u,v)$ and $k_2(u,v)$ satisfy respectively assumptions (4), (5) and (6). Then equation (26) is oscillatory provided the real numbers q_0 and e_0 satisfy:

$$\begin{cases} q_0 \ge \frac{3m}{K\lambda_0\beta\sin(\frac{\pi}{\beta})} & \text{if } \gamma = 1, \\ q_0^{\frac{1}{\gamma-1}} e_0 \ge 2(\gamma-1)\left(\frac{6m}{\gamma\left(K/2\right)^{1/\gamma}\lambda_0\beta\sin(\frac{\pi}{\beta})}\right)^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma > 1, \end{cases}$$

where the constants K and γ are from (4), the constant β is from (5), the constants α_1, α_2, and δ are from (5) and (6), and

$$\lambda_0 = \min\left\{\left(\frac{\alpha_1}{6m}\beta\sin\frac{\pi}{\beta}\right)^{\frac{1}{\beta-1}}, \left(\frac{\alpha_2}{12m}\beta\sin\frac{\pi}{\beta}\right)^{\frac{1}{\delta-1}}\right\}. \ \square$$

Consequently, from previous example we observe the following conclusion which is of the same kind as the result of Example 2.7.

Example 2.9. We consider the equation:

$$\big(k_1(x,x')\big)' + \cos(2t)k_2(x,x')x' + q_0\cos(2t)f(x) = e_0\cos(t),\ t \geq t_0 > 0, \tag{27}$$

where $f(u)$ satisfy (4) with $\gamma = 1$, and $k_1(u,v)$ and $k_2(u,v)$ satisfy (5) and (6) with $\alpha_1 = \alpha_2 = 1$ and $\beta = \delta = 2$. Then equation (27) is oscillatory provided $q_0 \geq 9/K$, where K is from (4). $\square$

In four previous examples, we consider equation (1) with the coefficient $\cos(2mt)$ and forcing term $\cos(t)$ or with the coefficient $\sin(2mt)$ and forcing term $\sin(t)$. Now, we present two examples for equation (1) with $\cos(t)$ on one hand side and $\sin(t)$ on other hand side and conversely.

Example 2.10. We consider the equation:

$$\big(k_1(x,x')\big)' + \cos(t)k_2(x,x')x' + q_0\cos(t)f(x) = e_0\sin(t),\ t \geq t_0 > 0, \tag{28}$$

where the functions $f(u)$, $k_1(u,v)$ and $k_2(u,v)$ satisfy respectively assumptions (4), (5) and (6). Then equation (28) is oscillatory provided the real numbers q_0 and e_0 satisfy:

$$\begin{cases} q_0 \geq \frac{24}{K\lambda_0\beta\sin(\frac{\pi}{\beta})} & \text{if } \gamma = 1, \\ q_0^{\frac{1}{\gamma-1}} e_0 \geq 2(\gamma-1)\left(\frac{12}{\gamma\big(K/2\big)^{1/\gamma}\lambda_0\beta\sin(\frac{\pi}{\beta})}\right)^{\frac{\gamma}{\gamma-1}} & \text{if } \gamma > 1, \end{cases} \tag{29}$$

where the constants K and γ are from (4), the constant β is from (5), the constants α_1, α_2, and δ are from (5) and (6), and

$$\lambda_0 = \min\left\{\left(\frac{\alpha_1}{12}\beta\sin\frac{\pi}{\beta}\right)^{\frac{1}{\beta-1}}, \left(\frac{\alpha_2}{24}\beta\sin\frac{\pi}{\beta}\right)^{\frac{1}{\delta-1}}\right\}. \ \square \tag{30}$$

Example 2.11. We consider the equation:

$$\big(k_1(x,x')\big)' + \sin(t)k_2(x,x')x' + q_0\sin(t)f(x) = e_0\cos(t),\ t \geq t_0 > 0, \tag{31}$$

where the functions $f(u)$, $k_1(u,v)$ and $k_2(u,v)$ satisfy respectively assumptions (4), (5) and (6). Then equation (31) is oscillatory provided the real numbers q_0 and e_0 satisfy (29) and (30). $\square$

The second type of the consequences of our main results are the following.

Corollary 2.12. *Let assumption* (4) *hold and let either* (5), (6) *or* $(1.5)_w$, (6) *or* (5), $(1.6)_w$ *hold. Let* $Q(t)$ *be given in* (7). *Let for any* $T \geq t_0$ *there exist* a_1, b_1, a_2, b_2, $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that* (2), (3) *and* (15) *hold. Let there exist positive constants* r_0, p_0, Q_0, ρ, μ, *and* ν *such that:*

$$r(t) \leq r_0 t^{-\rho},\ p(t) \geq p_0 t^{\mu} \quad \text{and} \quad Q(t) \geq Q_0 t^{\nu} \quad \text{on } [a_1, b_1] \cup [a_2, b_2].$$

Then equation (1) *is oscillatory.*

Example 2.13. Let the functions $f(u)$, $k_1(u, v)$ and $k_2(u, v)$ satisfy respectively assumptions (4), and either (5), (6) or $(1.5)_w$, (6). Let r_0, p_0, q_0, e_0, ρ, μ, σ_1, and σ_2 be arbitrary positive constants. Then equation:

$$\left(r_0 t^{-\rho} k_1(x, x')\right)' + p_0 t^{\mu} \sin(2t) k_2(x, x') x' + q_0 t^{\sigma_1} \sin(2t) f(x) = e_0 t^{\sigma_2} \sin(t),$$

is oscillatory. □

3 Proof of the main results

On an arbitrary interval $J \subseteq [t_0, \infty)$, a function $\overline{\omega} = \overline{\omega}(t)$, $\overline{\omega} \in C^1(J, \mathbb{R})$ is said to be a supersolution of the Riccati differential equation:

$$\omega'(t) = A_1(t)|\omega(t)|^{\beta} + A_2(t)|\omega(t)|^{\delta} + B(t),\ t \in J, \tag{32}$$

if $\overline{\omega}(t)$ satisfies the corresponding upper inequality: $\overline{\omega}'(t) \geq A_1(t)|\overline{\omega}(t)|^{\beta} + A_2(t)|\overline{\omega}(t)|^{\delta} + B(t)$, $t \in J$, where $A_1(t)$, $A_2(t)$ and $B(t)$ are three continuous functions determined in the following proposition.

Proposition 3.1. *Let assumptions* (4) *hold. Let for any* $T \geq t_0$ *there exist* a_1, b_1, a_2, b_2, $T \leq a_1 < b_1 \leq a_2 < b_2$ *such that* (2) *and* (3) *be satisfied. Let* $p(t) \geq 0$ *and* $q(t) \geq 0$ *on* $[a_1, b_1] \cup [a_2, b_2]$. *Let* $x(t)$ *be a nonoscillatory solution of the main equation* (1), *that is, let* $x(t) \neq 0$ *on* $[T, \infty)$ *for some* $T \geq t_0$. *If*

$$\overline{\omega}(t) = -\frac{\lambda r(t) k_1(x(t), x'(t))}{x(t)},\ t \geq T,\ \lambda > 0, \tag{33}$$

then $\overline{\omega}(t)$ *is a supersolution of the Riccati differential equation* (32) *on* $[a_1, b_1]$ *or* $[a_2, b_2]$ *with respect to:*

i) $A_1(t) = \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}$, $A_2(t) = \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)}$ *and* $B(t) = \lambda Q(t)$ *if* (5) *and* (6) *hold;*

ii) $A_1(t) = \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}$, $A_2(t) = 0$ *and* $B(t) = \lambda Q(t)$ *if* (5) *and* $(1.6)_w$ *hold;*

iii) $A_1(t) = 0$, $A_2(t) = \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)}$ *and* $B(t) = \lambda Q(t)$ *if* $(1.5)_w$ *and* (6) *hold;*

where α_1 *and* β *are from* (5), α_2 *and* δ *are from* (6), *the function* $Q(t)$ *is given in* (7).

Proof. Differentiating equality (33), and using (1), (4) and denoting by $x = x(t)$, $x' = x'(t)$, we easily compute:

$$\begin{aligned}\overline{\omega}'(t) &= \frac{\lambda r(t)}{x^2}k_1(x,x')x' - \frac{\big(\lambda r(t)k_1(x,x')\big)'}{x} \\ &= \frac{\lambda r(t)}{x^2}k_1(x,x')x' + \frac{\lambda p(t)}{x}k_2(x,x')x' + \frac{\lambda}{x}q(t)f(x) - \frac{\lambda}{x}e(t) \\ &\geq \frac{\lambda r(t)}{x^2}k_1(x,x')x' + \frac{\lambda p(t)}{x}k_2(x,x')x' + \lambda\big[Kq(t)|x|^{\gamma-1} + |e(t)||x|^{-1}\big],\end{aligned} \tag{34}$$

where $t \in J_{12}$ and either $J_{12} = [a_1, b_1]$ or $J_{12} = [a_2, b_2]$ such that $-e(t)/x = |e(t)||x|^{-1} > 0$ on J_{12}. Such an interval J_{12} exists because of assumptions (2). By the well-known inequality

$$\frac{1}{\mu_1}a + \frac{1}{\mu_2}b \geq a^{1/\mu_1}b^{1/\mu_2} \quad \text{for } a, b \geq 0,\ \mu_1, \mu_2 > 0 \text{ and } \tfrac{1}{\mu_1} + \tfrac{1}{\mu_2} = 1,$$

used in particular for $a = \gamma Kq(t)|x|^{\gamma-1}$, $b = \frac{\gamma}{\gamma-1}|e(t)||x|^{-1}$, $\mu_1 = \gamma$ and $\mu_2 = \gamma/(\gamma-1)$, from (34) it follows:

$$\overline{\omega}'(t) \geq \frac{\lambda r(t)}{x^2}k_1(x,x')x' + \frac{\lambda p(t)}{x}k_2(x,x')x' + \lambda Q(t),\ t \in J_{12}, \tag{35}$$

where $Q(t)$ is given in (7).

Next, the equality (33) can be rewriten in the form:

$$|k_1(x(t),x'(t))| = \frac{|x(t)|}{\lambda r(t)}|\overline{\omega}(t)|,\ t \geq T. \tag{36}$$

If assumptions (5) and (6) hold, then from (35) and (36) it follows:

$$\begin{aligned}\overline{\omega}'(t) &\geq \frac{\lambda r(t)}{|x|^\beta}\big[k_1(x,x')|x|^{\beta-2}x'\big] + \frac{\lambda p(t)}{|x|^\delta}\big[k_2(x,x')\mathrm{sgn}(x)|x|^{\delta-1}x'\big] + \lambda Q(t) \\ &\geq \frac{\alpha_1\lambda r(t)}{|x|^\beta}|k_1(x,x')|^\beta + \frac{\alpha_2\lambda p(t)}{|x|^\delta}|k_1(x,x')|^\delta + \lambda Q(t) \\ &= \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}|\overline{\omega}(t)|^\beta + \frac{\alpha_2 p(t)}{\lambda^{\delta-1}r^\delta(t)}|\overline{\omega}(t)|^\delta + \lambda Q(t),\ t \in J_{12},\end{aligned}$$

which proves the conclusion i) of this proposition.

If assumption (5) and $(1.6)_w$ hold, then from (35) and (36) it follows:

$$\begin{aligned}\overline{\omega}'(t) &\geq \frac{\lambda r(t)}{|x|^\beta}\big[k_1(x,x')|x|^{\beta-2}x'\big] + \frac{\lambda p(t)}{x^2}k_2(x,x')xx' + \lambda Q(t) \\ &\geq \frac{\alpha_1\lambda r(t)}{|x|^\beta}|k_1(x,x')|^\beta + \lambda Q(t) \\ &= \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}|\overline{\omega}(t)|^\beta + \lambda Q(t),\ t \in J_{12},\end{aligned}$$

which shows the conclusion ii) of the proposition.

And in the third case, if assumption $(1.5)_w$ and (6) hold, then from (35) and (36) we obtain:

$$\begin{aligned}\overline{\omega}'(t) &\geq \frac{\lambda r(t)}{x^2}k_1(x,x')x' + \frac{\lambda p(t)}{|x|^\delta}\left[k_2(x,x')\operatorname{sgn}(x)|x|^{\delta-1}x'\right] + \lambda Q(t)\\ &\geq \frac{\alpha_2\lambda p(t)}{|x|^\delta}|k_1(x,x')|^\delta + \lambda Q(t)\\ &= \frac{\alpha_2 p(t)}{\lambda^{\delta-1}r^\delta(t)}|\overline{\omega}(t)|^\delta + \lambda Q(t),\ t\in J_{12}.\end{aligned}$$

Thus, all conclusions of this propositions are shown. □

Next, a function $\underline{\omega} = \underline{\omega}(t)$, $\underline{\omega} \in C^1(J,\mathbb{R})$ is said to be a subsolution of the Riccati differential equation (32) if $\underline{\omega}(t)$ satisfies the corresponding lower inequality: $\underline{\omega}'(t) \leq A_1(t)|\underline{\omega}(t)|^\beta + A_2(t)|\underline{\omega}(t)|^\delta + B(t)$, $t\in J$, where $A_1(t)$, $A_2(t)$ and $B(t)$ are three continuous functions determined in the preceding proposition.

Proposition 3.2. *Let only once of conditions* (8), (9) *or* (10) *hold. Let* R_1 *and* R_2 *be two arbitrary real numbers. Then there are real numbers* $T_1^*\in[a_1,b_1)$ *and* $T_2^*\in[a_2,b_2)$, *and two functions* $\underline{\omega}_1\in C^1((a_1,T_1^*),\mathbb{R})\cap C([a_1,T_1^*),\mathbb{R})$ *and* $\underline{\omega}_2\in C^1((a_2,T_2^*),\mathbb{R})\cap C([a_2,T_2^*),\mathbb{R})$ *such that* $\underline{\omega}_i(t)$ *is a subsolution of the Riccati differential equation* (32) *on* $[a_i,T_i^*)$, $i\in\{1,2\}$, *and satisfies:*

$$\underline{\omega}_i(a_i) = R_i \quad \textit{and} \quad \lim_{t\to T_i^*}\underline{\omega}_i(t) = \infty,\ i\in\{1,2\}. \tag{37}$$

Proof. Let $\lambda > 0$ and let $C(t)$ be a function satisfying oscillatory condition (8). The other two cases for $C(t)$, that is, when condition (9) or (10) holds, can be analogously considered.

For $\beta > 1$ and two arbitrary real numbers R_1 and R_2, let s_1 and s_2 be two numbers such that

$$s_i\in(-\pi_\beta,\pi_\beta),\ \pi_\beta = \frac{\pi}{\beta\sin\frac{\pi}{\beta}} \quad \text{and} \quad y(s_i) = R_i,\ i\in\{1,2\}, \tag{38}$$

where $y = y(s)$ is an injective (increasing) function from $(-\pi_\beta,\pi_\beta)$ into $\mathbb{R}$ which satisfies:

$$\begin{cases} y\in C^1((-\pi_\beta,\pi_\beta),\mathbb{R}),\\ y'(s) = 1+|y(s)|^\beta, \quad s\in(-\pi_\beta,\pi_\beta),\\ y(0)=0 \quad \text{and} \quad y(\pi_\beta)=\infty. \end{cases} \tag{39}$$

If $\beta = 2$, then we can take $\pi_\beta = \pi/2$ and $y(s) = \tan(s)$, $s\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Such a function exists and it can be explicitly determined by the formula $y(s) = z^{-1}(s)$, where $z^{-1}(s)$ is the inverse function of $z = z(t)$ and z is a bijection from $\mathbb{R}$ on interval $(-z(\infty), z(\infty))$ defined by:

$$z(t) = \int_0^t \frac{1}{1+|\tau|^\beta} d\tau, \ t \in \mathbb{R} \quad \text{and} \quad z(\infty) = \pi_\beta,$$

where π_β is defined in (38).

From (8) and (38), it follows that the functions $V_1(t)$ and $V_2(t)$ defined by

$$V_i(t) = s_i + \frac{2\pi_\beta}{c_i} \int_{a_i}^t C(\tau)d\tau, \quad t \in [a_i, b_i], \ i \in \{1,2\}, \tag{40}$$

satisfy:

$$V_i(a_i) < \pi_\beta \quad \text{and} \quad V_i(b_i) > \pi_\beta, \ i \in \{1,2\}. \tag{41}$$

Since $C \in L^1((a_1,b_1) \cup (a_2,b_2), \mathbb{R})$, we conclude that $V_i \in AC([a_i,b_i], \mathbb{R})$ which together with (41) and $C(t)/c_i \geq 0$ ensures the existence of two points $T_1^* \in (a_1,b_1)$ and $T_2^* \in (a_2,b_2)$ such that

$$V_i(T_i^*) = \pi_\beta \quad \text{and} \quad V_i : [a_i, T_i^*] \to [s_i, \pi_\beta] \subset (-\pi_\beta, \pi_\beta], \ i \in \{1,2\}. \tag{42}$$

Hence, according to (38), (39) and (42) we conclude that the following two functions $\underline{\omega}_1(t)$ and $\underline{\omega}_2(t)$,

$$\underline{\omega}_i(t) = y(V_i(t)), \ t \in [a_i, T_i^*), \ i \in \{1,2\}, \tag{43}$$

are well-defined and for $i \in \{1,2\}$ satisfy

$$\underline{\omega}_i(a_i) = y(V_i(a_i)) = y(s_i) = R_i \quad \text{and} \quad \lim_{t \to T_i^*} \underline{\omega}_i(t) = y(V_i(T_i^*)) = y(\pi_\beta) = \infty,$$

which proves desired statement (37). Next, from (40) we obtain $V_i'(t) = \frac{2\pi_\beta}{c_i} C(t)$ and hence from (8), (39) and (43) we have for $t \in [a_i, T_i^*)$:

$$\begin{aligned}
\underline{\omega}_i'(t) = y'(V_i(t))V_i'(t) &= \frac{2\pi_\beta}{c_i} C(t)(1 + |\underline{\omega}_i(t)|^\beta) \\
&\leq \frac{2\pi_\beta}{c_i} C(t)\big(|\underline{\omega}_i(t)|^\beta + |\underline{\omega}_i(t)|^\delta + 1\big) \\
&= \frac{2\pi_\beta}{c_i} C(t)|\underline{\omega}_i(t)|^\beta + \frac{2\pi_\beta}{c_i} C(t)|\underline{\omega}_i(t)|^\delta + \frac{2\pi_\beta}{c_i} C(t) \\
&\leq \frac{\alpha_1}{(\lambda r(t))^{\beta-1}} |\underline{\omega}_i(t)|^\beta + \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^\delta(t)} |\underline{\omega}_i(t)|^\delta + \lambda Q(t).
\end{aligned}$$

Thus, according to previous consideration, the conclusion of this proposition is shown, when $C(t)$ satisfies condition (8). Other cases for $C(t)$, that is, when conditions (9) or (10) hold, it can be analogously considered. □

Proof of Theorem 1.1. We start by a pointwise comparison principle stated in [9, Lemma 4.1]. About the pointwise comparison principles in other types of integral and differential equations, we refer reader to [19, 20, 8, 13] and references therein.

Lemma 3.3. *Let T_0 and T^* be two arbitrary real numbers such that $T_0 < T^*$. Let $\tilde{\varphi}(t)$ and $\tilde{\psi}(t)$, $\tilde{\varphi}, \tilde{\psi} \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$, be two functions satisfying:*

$$\tilde{\varphi}' \leq h(t, \tilde{\varphi}) \quad \text{and} \quad \tilde{\psi}' \geq h(t, \tilde{\psi}),\ t \in (T_0, T^*), \tag{44}$$

where $h(t, u)$ is a locally Lipschitz function in the second variable. Then we have:

$$\tilde{\varphi}(T_0) \leq \tilde{\psi}(T_0) \quad \text{implies} \quad \tilde{\varphi}(t) \leq \tilde{\psi}(t) \text{ for all } t \in [T_0, T^*). \tag{45}$$

Let us remark that, as usually, a function $h : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz in the second variable if for any interval $[a, b] \subset [t_0, \infty)$ and $M > 0$, there is a function $L = L(t)$, $L \in C^1((a, b), \mathbb{R}_+)$ depending on $[a, b]$, M and h such that for all $t \in [a, b]$ and $u_1, u_2 \in \mathbb{R}$, $|u_1| \leq M$, $|u_2| \leq M$,

$$|h(t, u_1) - h(t, u_2)| \leq L(t)|u_1 - u_2|.$$

Proposition 3.4. *For every two sub- and supersolution of equation* (32), *the statement* (45) *is fulfilled on arbitrary interval $[T_0, T^*)$, $t_0 \leq T_0 < T^*$.*

Proof. Let $\varphi(t)$ and $\psi(t)$, $\varphi, \psi \in C^1((T_0, T^*), \mathbb{R}) \cap C([T_0, T^*), \mathbb{R})$, be respectively a sub- and super solution of equation (32), that is, they satisfy (44), where $h(t, u) := A_1(t)|u|^\beta + A_2(t)|u|^\delta + B(t)$. Since $A_1(t)$, $A_2(t)$ and $B(t)$ are continuous functions, it is not difficult to check that such a $h(t, u)$ is a locally Lipschitz function in the second variable. Hence this proposition follows from Lemma 3.3. □

Next, if equation (1) does not oscillatory, then there is at least one nonoscillatory solution $x(t)$ of (1), so that $x(t) \neq 0$ on $[T, \infty)$ for some $T \geq t_0$. Now, by Proposition 3.1 the function $\overline{\omega} \in C([a_1, b_1), \mathbb{R})$ defined by (33) is a supersolution of the Riccati differential equation (32) on the whole interval J_{12} (we can take for instance $J_{12} = [a_1, b_1]$, the other case $J_{12} = [a_2, b_2]$ can be similarly considered). On the other hand, by Proposition 3.2 we obtain a number $T_1^* \in [a_1, b_1)$ and a function $\underline{\omega}_1 \in C([a_1, T_1^*), \mathbb{R})$ such that $\underline{\omega}_1(t)$ is a subsolution of equation (32) on $[a_1, T_1^*)$, and satisfies (37). Now applying Proposition 3.4 on these two functions $\underline{\omega}_1(t)$ and $\overline{\omega}(t)$ on J_{12} we conclude that $\infty = \lim_{t\to T_1^*} \underline{\omega}_1(t) \le \lim_{t\to T_1^*} \overline{\omega}(t)$, which contradicts the fact that $\overline{\omega} \in C([a_1, b_1), \mathbb{R})$. In conclusion, there is no any nonoscillatory solution of equation (1). □

Proof of Theorem 1.3. The second and third steps from the proof of Theorem 1.1 can be used here in exactly the same way. But, in the first step, instead of Proposition 3.1 we use the next fact: if $x(t)$ is a nonoscillatory solution of equation (1), then according to assumptions (11), (12) and (13), the function $\overline{\omega}(t)$ defined by:

$$\overline{\omega}(t) = -\frac{\lambda r(t) k_1(x(t), x'(t))}{f(x(t))},\ t \ge T,\ \lambda > 0,$$

is a supersolution of the Riccati differential equation (32) with the coefficients:

$$A_1(t) = \frac{\alpha_1 K}{(\lambda r(t))^{\beta-1}},\quad A_2(t) = \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)} \quad \text{and} \quad B(t) = \lambda q(t).$$

The rest of this proof is left to the reader taking into account appropriate arguments from the proof of Theorem 1.1. □

Proofs of Corollaries 2.4, 2.5 and 2.12. Immediately from Theorems 1.1 and 1.3 in particular for $C(t) \equiv 1$. □

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