Some New Inequalities for Symmetric Matrix with Spectral Decomposition

Benxiu Li
School of Mathematics and Statistics, Chongqing Three Gorges University
Wanzhou, Chongqing, 404000, P. R. China

Feixiang Chen
School of Mathematics and Statistics, Chongqing Three Gorges University
Wanzhou, Chongqing, 404000, P. R. China

Abstract

We present some new inequalities for symmetric matrix with spectral decomposition. We obtain the inequalities with Frobenius norm of matrix A and its spectral radius.

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1 Introduction

Throughout this paper, we use the following notations: $S^n$ The set of all $n \times n$ symmetric matrices
$S^n_+$ The set of all $n \times n$ symmetric positive semidefinite matrices
$S^n_{++}$ The set of all $n \times n$ symmetric positive definite matrices
$\lambda_i(Q)$ The eigenvalues of $Q \in S^n$, $i = 1, 2, \ldots, n$
$\lambda_{\text{min}}(Q)$ The smallest eigenvalue of $Q \in S^n$
$\lambda_{\text{max}}(Q)$ The largest eigenvalue of $Q \in S^n$
$\Lambda(Q)$ The diagonal matrix with all eigenvalues $Q$ as diagonal elements

Let $A$ be any $m \times n$ matrix. The Frobenius norm of matrix $A$ is
\[ \|A\|_F = \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{1/2}. \]

It is also equal to the square root of the matrix trace of \( AA^T \), that is
\[ \|A\|_F = \sqrt{tr(\AA)} \]

Let \( \| \cdot \| \) be a vector norm on \( \mathbb{R}^n \). Define operator norm or spectral norm \( \| \cdot \| \) on \( M_n \) by
\[ \|A\| \equiv \max_{\|x\|=1} \|Ax\| = \sqrt{\lambda_{\max}(A^TA)}. \]

Let \( M \) be a symmetric real matrix, i.e., \( M \in S^n \), with the spectral decomposition \( M = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T \), where \( \Lambda \) is a diagonal matrix with all the eigenvalues of \( M \) along its diagonal, \( Q \) is an orthonormal matrix, i.e., \( QQ^T = I \), and each column \( q_i \) of \( Q \) is an eigenvector of \( M \) corresponding to the eigenvalue \( \lambda_i \). Then we define the positive part \( M^+ \) and the negative part \( M^- \) of \( M \) as
\[ M^+ := \sum_{\lambda_i \geq 0} \lambda_i q_i q_i^T, \quad M^- := \sum_{\lambda_i \leq 0} \lambda_i q_i q_i^T. \tag{1} \]

Apparently, \( M = M^++M^- \), \((-M)^+ = -M^- \), \((-M)^- = -M^+ \), where \( M \succeq 0 \), \(-M^- \succeq 0 \).

We present some interesting properties of the positive and the negative parts of a symmetric matrix which play a crucial role in the matrix analysis.

## 2 Lemmas

**Lemma 2.1** (see, e.g., [3]). For all \( E \in S^n \), we have:
\[ \lambda_{\max}(E) = \max_{\|u\|=1} u^T E u. \tag{2} \]
\[ \lambda_{\min}(E) = \min_{\|u\|=1} u^T E u. \tag{3} \]
\[ \|E\| = \max_{i=1,\ldots,n} |\lambda_i(E)|. \tag{4} \]
\[ \|E\|_F^2 = \sum_{i=1}^{n} [\lambda_i(E)]^2. \tag{5} \]

**Lemma 2.2** For all \( A \in \mathbb{R}^{n \times n} \), the following relations hold:
\[ \sum_{i=1}^{n} |\lambda_i(A)|^2 \leq \|A\|_F^2 = \|A^T\|_F^2; \tag{6} \]
Lemma 2.3 (see, e.g., [3]). Suppose that $W \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. then for any $E \in S^n$, the following relations hold:

$$
\lambda_{\text{max}}(E) \leq \frac{1}{2} \lambda_{\text{max}}(WEW^{-1} + (WEW^{-1})^T).
$$

(7)

$$
\lambda_{\text{min}}(E) \geq \frac{1}{2} \lambda_{\text{min}}(WEW^{-1} + (WEW^{-1})^T).
$$

(8)

Lemma 2.4 (see, e.g., [3]). Suppose that $W \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. then for any $E \in S^n$, the following relations hold:

$$
\|E\| \leq \frac{1}{2} \|WEW^{-1} + (WEW^{-1})^T\|.
$$

(9)

$$
\|E\|_F \leq \frac{1}{2} \|WEW^{-1} + (WEW^{-1})^T\|_F.
$$

(10)

Lemma 2.5 (see, e.g., [4]). Suppose that $U, V \in S^n$, then the following statements holds:

$$
\|(U + V)^+\|_F \leq \|U^+ + V^+\|_F \leq \|U^+\|_F + \|V^+\|_F
$$

Lemma 2.6 (see, e.g., [4]). Suppose that $M \in S^n$ and $W$ be a nonsingular matrix. Then the following statements holds:

$$
\|M^+\|_F = \|(WMW^{-1})^+\|_F
$$

3 Results and Discussion

Lemma 3.1 Suppose that $U, V \in S^n$, then the following statements holds:

$$
\|(U + V)^-\|_F \leq \|U^- + V^-\|_F \leq \|U^-\| + \|V^-\|_F
$$

(11)

$$
\|(U + V)^+\| \leq \|U^+ + V^+\| \leq \|U^+\| + \|V^+\|
$$

(12)

$$
\|(U + V)^-\| \leq \|U^- + V^-\| \leq \|U^-\| + \|V^-\|
$$

(13)

Proof. The second inequality is straightforward. We show only the first one. Note that

$$
U = U^+ + U^- = U^+ + \sum_{\lambda_i(U) \leq 0} \lambda_i(U)q_i(U)q_i(U)^T
$$

and

$$
V = V^+ + V^- = V^+ + \sum_{\lambda_i(V) \leq 0} \lambda_i(V)q_i(V)q_i(V)^T
$$
According to Theorem 8.1.5 in Golub and Van Loan [2], we obtain
\[ \lambda_i(U^- + V^-) \leq \lambda_i(U + V) \leq \lambda_i(U^+ + V^+) \]
for \( i = 1, \ldots, n \). Let \( I_+ \) denote the index set such that \( I_+ := \{ i | \lambda_i(U + V) \geq 0 \} \) and \( I_- \) denote the index set such that \( I_- := \{ i | \lambda_i(U + V) \leq 0 \} \), respectively. Then
\[
\| (U + V)^- \|_F^2 = \sum_{i \in I_-} \lambda_i^2(U^- + V^-) \leq \sum_{i \in I_+} \lambda_i^2(U + V) \leq \| U^- + V^- \|_F^2
\]
\[
\| (U + V)_+ \| = \max_{i \in I_+} \lambda_i(U + V) \leq \max_{i \in I_+} \lambda_i(U^+ + V^+) \leq \| U^+ + V^+ \|
\]
\[
\| (U + V)^- \| = \max_{i \in I_-} | \lambda_i(U + V) | \leq \max_{i \in I_-} | \lambda_i(U^- + V^-) | \leq \| U^- + V^- \|
\]
which completes the proof.

The next lemma reveals that a similarity transformation preserves the Frobenius norm over the positive part and the negative part of a symmetric matrix.

**Lemma 3.2** Suppose that \( M \in S^n \) and \( W \) be a nonsingular matrix. Then the following statements holds:
\[
\| M^+ \| = \| (WMW^{-1})^+ \| \tag{14}
\]
\[
\| M^- \| = \| (WMW^{-1})^- \| \tag{15}
\]

**Proof.** The equalities are straightforward from the similarity of \( M \) and \( WMW^{-1} \).

**Theorem 3.3** Suppose that \( M \in S^n \) and \( W \) be a nonsingular matrix. Then the following statements holds:
\[
\| M^+ \| \leq \frac{1}{2} \| [WMW^{-1} + (WMW^{-1})^T]^+ \| \tag{16}
\]
\[
\| M^- \| \leq \frac{1}{2} \| [WMW^{-1} + (WMW^{-1})^T]^- \| \tag{17}
\]

**Proof.** To prove the first equality, we first consider the situation that \( M \) is negative semidefinite, \( M^+ = 0 \), the inequality is hold. When \( \lambda_{\max}(M) > 0 \), using the inequality \( \lambda_{\max}(M) \leq \frac{1}{2} \lambda_{\max}[WMW^{-1} + (WMW^{-1})^T] \), we obtain
\[
\| M^+ \| = \lambda_{\max}(M) \\
\leq \frac{1}{2} \lambda_{\max}[WMW^{-1} + (WMW^{-1})^T] \\
= \frac{1}{2} \| [WMW^{-1} + (WMW^{-1})^T]^+ \|
\]
Then we get the equality (16).
Now let us consider the second equality. When $\lambda_{\min}(M) \geq 0$, then $M^{-} = 0$, the inequality is hold. When $\lambda_{\min}(M) < 0$, using the inequality $\lambda_{\min}(M) \geq \frac{1}{2}\lambda_{\min}[WMW^{-1} + (WMW^{-1})^T]$, we obtain

$$
\|M^{-}\| = |\lambda_{\min}(M)| \\
\leq \frac{1}{2}|\lambda_{\min}[WMW^{-1} + (WMW^{-1})^T]| \\
= \frac{1}{2}\|WMW^{-1} + (WMW^{-1})^T\|^{-1}
$$

Then we obtain the equality (17).

References


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