Comments on Convergence Rates of Mann and Ishikawa Iterative Schemes for Generalized Contractive Operators

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Abstract

In [1], Olaleru made the claim that Mann iteration converges faster than Ishikawa iteration when applied to generalized contractive operators. By providing an example we prove that this claim is false.

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1. Introduction and Preliminaries

In the recent years, fixed points of operators have been approximated by various authors[1, 2, 4, 7] using different iterative schemes. The following iteration schemes are now well known:

\[ u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n, \quad (1) \]

where \{ \alpha_n \} is a sequence of positive numbers in [0,1], is called Mann iteration [8].

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \]
\[ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad (2) \]
where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences of positive numbers in \([0,1]\), is called Ishikawa iteration [5].

It is obvious that Mann iterative can be obtained from Ishikawa iterative scheme after putting \( \beta_n = 0 \).

**Definition 1** [1] Let \( T \) be a mapping of a metric space \((X, d)\) into itself. A mapping \( T \) will be called a generalized contractive operator if for some \( 0 \leq k < 1 \) and all \( x, y \in X \),

\[
d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\}
\]

Fixed point iterative schemes are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative schemes. In computational mathematics, it is of vital interest to know which of the given iterative scheme converges faster to a desired solution, commonly known as rate of convergence. In [6] Berinde showed that Picard iteration is faster than Mann iteration for quasi-contractive operators satisfying the following contractive definition: there exist \( a \in [0, 1) \) and a monotone increasing function \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \varphi(0) = 0 \), such that

\[
\|Tx - Ty\| \leq \varphi(\|x - Tx\|) + a \|x - y\|
\]

and following definition:

**Definition 2** [6, 7] Let \( \{u_n\} \) and \( \{v_n\} \) be two fixed point iteration procedures that converges to the same fixed point \( p \) on a normed space \( X \) such that the error estimates

\[
\|u_n - p\| \leq a_n, \tag{5}
\]

and

\[
\|v_n - p\| \leq b_n, \tag{6}
\]

are available, where \( \{a_n\} \) and \( \{b_n\} \) are two sequences of positive numbers (converging to zero). If \( \{a_n\} \) converge faster than \( \{b_n\} \), then we say that \( \{u_n\} \) converge faster to \( p \) than \( \{v_n\} \).

In [1, Theorem 2], Olaleru proved the following by using Definition 2:

**Theorem 3.** Let \( K \) be a nonempty closed convex subset of a Banach space \( X \) and \( T: K \to K \) a generalized contractive map (3). Let \( \{x_n\} \) and \( \{y_n\} \) be the Mann and Ishikawa iterations respectively defined by (I) and (M) for \( x_0, y_0 \in K \) with \( \{\alpha_n\} \) and \( \{\beta_n\} \) real sequences such that \( 0 \leq \alpha_n, \beta_n \leq 1 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then Mann iteration converges to the fixed point of \( T \) faster than Ishikawa iteration.
Comments on convergence rates

We use the following definition.

**Definition 4** [2] Suppose \( \{a_n\} \) and \( \{b_n\} \) are two real convergent sequences with limits \( a \) and \( b \), respectively. Then \( \{a_n\} \) is said to converge faster than \( \{b_n\} \) if

\[
\lim_{n \to \infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.
\]

2. Main Result

The example given below satisfies all the conditions of Theorem 3 but still Ishikawa iterative scheme converges faster than Mann iterative scheme.

**Example 2.1.** Let \( T : [0,1] \to [0,1] : x = \frac{x}{4} \), \( \alpha_n = \beta_n = 0 \) for \( n = 1, 2 \ldots 15 \) and \( \alpha_n = \beta_n = \frac{4}{\sqrt{n}} \) for \( n \geq 16 \). It is easy to see that for \( k = 0.7 \), \( T \) is a generalized contractive operator satisfying (3) with a unique fixed point 0. Also for \( n \geq 16 \),

\[
\frac{4}{\sqrt{n}} \in [0,1] \text{ and } \sum_{n=16}^{\infty} \frac{4}{\sqrt{n}} = \infty.
\]

Thus all the conditions of Theorem 3 are satisfied. But we show that Ishikawa iterative scheme converges faster than Mann iterative scheme.

Let \( u_0 = x_0 \neq 0 \). Then using Mann and Ishikawa iterative schemes, respectively, we have

\[
u_{n+1} = (1 - \alpha_n)u_n + \alpha_nTu_n
\]

\[
= (1 - \frac{4}{\sqrt{n}})u_n + \frac{4}{\sqrt{n}} \frac{u_n}{4}
\]

\[
= (1 - \frac{3}{\sqrt{n}})u_n
\]

\[
= \prod_{i=16}^{n} (1 - \frac{3}{\sqrt{i}})u_0
\]

and
\[ y_{n+1} = (1 - \alpha_n) x_n + \alpha_n T ((1 - \beta_n) x_n + \beta_n T x_n), \]
\[ = (1 - \frac{4}{\sqrt{n}}) x_n + \frac{4}{\sqrt{n}} \left( (1 - \frac{4}{\sqrt{n}}) x_n + \frac{4}{\sqrt{n}} \right) x_n \]
\[ = (1 - \frac{3}{\sqrt{n}}) x_n \]
\[ = \prod_{i=1}^{n} \left( 1 - \frac{3}{\sqrt{i}} \right) x_0 \]

Now, consider
\[
\left| \frac{y_{n+1}}{u_{n+1}} \right| = \left| \prod_{i=1}^{n} \left( 1 - \frac{3}{\sqrt{i}} \right) \frac{x_0}{x_0} \right| = \left| \prod_{i=1}^{n} \left( 1 - \frac{3}{\sqrt{i}} \right) \right| = \left| \prod_{i=1}^{n} \left( 1 - \frac{3}{i - 3\sqrt{i}} \right) \right| \]

It is easy to see that
\[
0 \leq \lim_{n \to \infty} \prod_{i=1}^{n} \left( 1 - \frac{3}{i - 3\sqrt{i}} \right) \leq \lim_{n \to \infty} \prod_{i=1}^{n} \left( 1 - \frac{1}{i} \right) = \lim_{n \to \infty} \frac{15}{n} = 0. \]

Hence, \( \lim_{n \to \infty} \left| \frac{y_{n+1}}{u_{n+1}} \right| = 0. \)

Therefore, by Definition 4, Ishikawa iterative scheme converges faster than Mann iterative scheme to the fixed point 0 of \( T. \)
Hence Theorem 3 is not consistent.

The error occur because the Definition 2 used by Olaleru is not consistent as shown by Popescu [3] by providing the following example:

**Example 2.2** If we have \( a_n = \frac{1}{n^2}, \ b_n = \frac{1}{n} \), then \( \{u_n\} \) converges faster than \( \{v_n\}. \)
But if we take \( b_n = \frac{1}{n^2} \), (supposing that (1.6) is still available) we obtain that \( \{v_n\} \) converges faster than \( \{u_n\}. \)
References


2. N. Hussain, R. Chugh, V. Kumar and A. Rafiq, On The Rate of Convergence of Kirk Type Iterative Schemes, Journal of Applied Mathematics, Vol 2012, 22 pages


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