Fekete-Szegö Estimation Problem
for Some Subclasses of Analytic Functions

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Abstract. In this present investigation, the authors obtain Fekete-Szegö’s inequality for certain normalized analytic functions $f(z)$ defined on the open unit disk for which $zf'(z) - (1 - \lambda) f(z) + \lambda z f'(z)$ $(0 \leq \lambda < 1)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö’s inequality for a class of functions defined through fractional derivatives is obtained. The Motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra.

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1. Introduction

Let $A$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} ||z| < 1\})$$

and $S$ be the subclass of $A$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in S$ for which

$$zf'(z) f(z) \prec \phi(z), \quad (z \in \Delta)$$

and $C(\phi)$ be the class of functions in $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [10]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the paper by Srivatsava et al. [7].

In this paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_\lambda(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M^\alpha_\lambda(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivatsava and Mishra [6].

**Definition 1.1.** Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk $\Delta$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is in the class $M_\lambda(\phi)$ if

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} \prec \phi(z) \quad (0 \leq \lambda < 1).$$

For fixed $g \in A$, we define the class $M^g_\lambda(\phi)$ to be the class of functions $f \in A$ for which $(f \ast g) \in M_\lambda(\phi)$.

To prove our main result, we need the following.
Lemma 1.1. [10] If $p_1(z) = 1 + c_1z + c_2z^2 + \ldots$ is an analytic function with positive real part in $\Delta$, then

$$|c_2 - vc_1^2| \leq \begin{cases} 
-4v + 2 & \text{if } v \leq 0; \\
2 & \text{if } 0 \leq v \leq 1; \\
4v - 2 & \text{if } v \geq 1.
\end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$.

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad \left(0 < v \leq \frac{1}{2}\right)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad \left(\frac{1}{2} < v \leq 1\right).$$

2. COEFFICIENT PROBLEM

Our main result is the following.

Theorem 2.1. Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots$. If $f(z)$ is given by (1.1.1) belongs to $M_\lambda(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{B_2}{2(1-\lambda)} - \frac{\mu}{(1-\lambda)^2}B_1^2 + \frac{(1+\lambda)}{2(1-\lambda)^2}B_1^2 & \text{if } \mu \leq \sigma_1; \\
\frac{B_1}{2(1-\lambda)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
-\frac{B_2}{2(1-\lambda)} + \frac{\mu}{(1-\lambda)^2}B_1^2 - \frac{(1+\lambda)}{2(1-\lambda)^2}B_1^2 & \text{if } \mu \geq \sigma_2,
\end{cases}$$

where

$$\sigma_1 := \frac{(1-\lambda)(B_2 - B_1) + (1+\lambda)B_1^2}{2B_1^2},$$

$$\sigma_2 := \frac{(1-\lambda)(B_2 + B_1) + (1+\lambda)B_1^2}{2B_1^2}.$$

The result is sharp.
Proof. For $f(z) \in M_\lambda(\phi)$, let

$$p(z) := \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} = 1 + b_1 z + b_2 z^2 + \ldots$$ \hspace{1cm} (2.2.1)

From (2.2.1), we obtain

$$b_1 = (1 - \lambda)a_2 \text{ and } b_2 + (1 - \lambda)(1 + \lambda)a_2^2 = 2a_3(1 - \lambda).$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic and has a positive real part in $\Delta$. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right)$$ \hspace{1cm} (2.2.2)

and from this equation (2.2.2), we obtain

$$b_1 = \frac{1}{2}B_1c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4(1 - \lambda)}\{c_2 - vc_1^2\},$$ \hspace{1cm} (2.2.3)

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{2\mu - (1 + \lambda)}{(1 - \lambda)}B_1\right].$$

Our result now follows by an application of Lemma 1.1. To show that the bounds are sharp, we define the functions $K_\lambda^{\phi_n}(n = 2, 3, 4, \ldots)$ by

$$z \left[K_\lambda^{\phi_n}\right]'(z) = \phi\left(z^{n-1}\right), K_\lambda^{\phi_n}(0) = 0 = [K_\lambda^{\phi_n}]'(0) - 1$$

and the function $F_\lambda^\alpha$ and $G_\lambda^\alpha$ ($0 \leq \alpha \leq 1$) by

$$z \left[F_\lambda^\alpha\right]'(z) = \phi\left(z(1 + \alpha\right), F_\lambda^\alpha(0) = 0 = (F_\lambda^\alpha)'(0) - 1$$

and

$$z \left[G_\lambda^\alpha\right]'(z) = \phi\left(\frac{-z(1 + \alpha)}{1 + \alpha z}\right), G_\lambda^\alpha(0) = 0 = (G_\lambda^\alpha)'(0).$$

Clearly the functions $K_\lambda^{\phi_n}, F_\lambda^\alpha, G_\lambda^\alpha \in M_\lambda(\phi)$. Also write

$$K_\lambda := K_\lambda^{\phi_2},$$
If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f$ is $K^{\phi}_\lambda$ or one of its rotations.

When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if $f$ is $K^{\phi_3}_\lambda$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if $f$ is $G^{\alpha}_\lambda$ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if $f$ is $G^{\alpha}_\lambda$ or one of its rotations.

**Remark 2.1.** If $\sigma_1 \leq \mu \leq \sigma_2$, then in view of Lemma 1.1, Theorem 2.1 can be improved. Let

$$\sigma_3 = \frac{B_2 (1 - \lambda) + (1 + \lambda) B_1^2}{2 B_1^2}.$$  

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1 - \lambda)}{2 B_1^2} \left[ (B_1 - B_2) + (2 \mu - 1 - \lambda) B_1^2 \right] |a_2|^2 \leq \frac{B_1}{2(1 - \lambda)}.$$  

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1 - \lambda)}{2 B_1^2} \left[ (B_1 + B_2) - (2 \mu - 1 - \lambda) B_1^2 \right] |a_2|^2 \leq \frac{B_1}{2(1 - \lambda)}.$$  

3. **APPLICATIONS TO FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVES**

In order to introduce the class $M^{\alpha}_\lambda(\phi)$, we need the following.

**Definition 3.1.** (see [3, 4], see also [8, 9])

Let $f(z)$ be analytic in a simply connected region of the $z$-plane containing the origin. The fractional derivative of $f$ of order $\alpha$ is defined by

$$D^\alpha_z f(z) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^\alpha} d\zeta \quad (0 \leq \alpha < 1),$$

where the multiplicity of $(z - \zeta)^\alpha$ is removed by requiring that $\log(z - \zeta)$ is real for $z - \zeta > 0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^\alpha : A \to A$ defined by

$$(\Omega^\alpha f)(z) = \Gamma(2 - \alpha) z^{\alpha} D^\alpha_z f(z), \quad (\alpha \neq 2, 3, 4, \ldots).$$

The class $M^{\alpha}_\lambda(\phi)$ consists of functions $f \in A$ for which $\Omega^\alpha f \in M^{\alpha}_\lambda(\phi)$. Note that $M^{\alpha}_0(\phi) \equiv S^\alpha(\phi)$ and $M^{\alpha}_\lambda(\phi)$ is the special case of the class $M^{\alpha}_\lambda(\phi)$ when

$$(3.3.1) \quad g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n + 1) \Gamma(2 - \alpha)}{\Gamma(n + 1 - \alpha)} z^n,$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0).$$
Since
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_\lambda^g(\phi) \]
if and only if
\[ (f * g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_\lambda(\phi), \]
we obtain the coefficient estimate for functions in the class \( M_\lambda^g(\phi) \), from the corresponding estimate for functions in the class \( M_\lambda(\phi) \). Applying Theorem 2.1 for the function
\[ (f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \ldots, \]
we get the following Theorem 3.1, after an obvious change of the parameter \( \mu \);

**Theorem 3.1.** Let the function \( \phi(z) \) be given by
\[ \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots. \]
If \( f(z) \) given by (1.1.1) belongs to \( M_\lambda^g(\phi) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{g_3} \left[ \frac{B_2}{2(1-\lambda)} - \frac{\mu g_3}{(1-\lambda)^2 g_2} B_1^2 + \frac{(1+\lambda)}{2(1-\lambda)^2} B_1^2 \right] & \text{if } \mu \leq \sigma_1; \\
\frac{1}{g_3} \left[ \frac{B_1}{2(1-\lambda)} \right] & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{1}{g_3} \left[ \frac{-B_2}{2(1-\lambda)} + \frac{\mu g_3}{(1-\lambda)^2 g_2} B_1^2 - \frac{(1+\lambda)}{2(1-\lambda)^2} B_1^2 \right] & \text{if } \mu \geq \sigma_2,
\end{cases}
\]
where
\[
\sigma_1 := \frac{g_2^2}{g_3} \left[ \frac{(1-\lambda)(B_2 - B_1) + (1+\lambda)B_1^2}{2B_1^2} \right],
\]
\[
\sigma_2 := \frac{g_3^2}{g_3} \left[ \frac{(1-\lambda)(B_2 + B_1) + (1+\lambda)B_1^2}{2B_1^2} \right].
\]
The result is sharp.

Since, \( (\Omega^\alpha f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, \)
we have,
\[ g_2 := \frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)} = \frac{2}{(3-\alpha)} \]
and
\[ g_3 := \frac{\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)} = \frac{6}{(2-\alpha)(3-\alpha)}. \]
For \( g_2 \) and \( g_3 \) given by (3.3.2) and (3.3.3), Theorem 3.1 reduces to the following:
**Theorem 3.2.** Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f(z)$ given by (1.1.1) belongs to $M_\lambda^\alpha(\phi)$, then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{(2 - \alpha)(3 - \alpha)\gamma}{6} & \text{if } \mu \leq \sigma_1; \\
\frac{(2 - \alpha)(3 - \alpha)B_1}{6(1 - \lambda)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{- (2 - \alpha)(3 - \alpha)\gamma}{6} & \text{if } \mu \geq \sigma_2,
\end{cases}
\]
where
\[
\gamma := \frac{B_2}{2(1 - \lambda)} - \frac{3(2 - \alpha)}{2(3 - \alpha)(1 - \lambda)^2} B_1^2 + \frac{(1 + \lambda)}{2(1 - \lambda)^2} B_1^2,
\]
\[
\sigma_1 := \frac{2(3 - \alpha)}{3} \left[ \frac{(1 - \lambda)(B_2 - B_1) + (1 + \lambda)B_1^2}{2B_1^2} \right],
\]
\[
\sigma_2 := \frac{2(3 - \alpha)}{3} \left[ \frac{(1 - \lambda)(B_2 + B_1) + (1 + \lambda)B_1^2}{2B_1^2} \right].
\]
The result is sharp.

**Remark 3.1.** When $\lambda = 0, B_1 = \frac{8}{\pi^2}$ and $B_2 = \frac{16}{3\pi^2}$, the above Theorem 3.1 reduces to a result of Srivastava and Mishra [6, Theorem 8, p.64] for a class of functions for which $\Omega^\alpha f(z)$ is a parabolic starlike function [2, 5].

**References**


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