Analysis of an Economic Growth Model with Variable Carrying Capacity

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Abstract

In this paper, a kind of nonlinear delayed differential equations which can describe an economic model is considered. In this economy, population is modelled according to a logistically variable carrying capacity and capital’s accumulation has a time delay. By choosing time delay as a bifurcation parameter, it is proved that the system loses stability and a Hopf bifurcation occurs when the time delay passes through critical values.

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1 Introduction

The logistic equation, advanced by Verhulst [19], has been the workhorse model for describing the evolution of various social, biological, and economic systems. Mathematically, it is described by

$$\dot{N} = \gamma N \left(1 - \frac{N}{\kappa}\right),$$

where $N$ is the population size, the coefficient $\gamma$ is a reproduction rate and $\kappa$ is the carrying capacity. All known variants of the logistic equation describe either a single-step evolution, called the S-curve, or an oscillatory behavior.
around a constant level. However, the development of many complex systems consists not just of a single step, but by multistep growth phases, where a period of fast growth is followed by a lasting period of stagnation or saturation, which is itself followed by another fast growth regime, and so on. To capture the previously described phenomenology and much more, the constant $\kappa$ in Eq. (1) is replaced with a nonconstant carrying capacity, which obviously increases the complexity of the model’s behavior. Numerous models of dynamic carrying capacity have been proposed and studied over the past years. For example, Banks [1994] describes models where the carrying capacity varies sinusoidally, exponentially and linearly. Coleman [5] studies general forms of (1) with variable carrying capacity and determined global mathematical properties. Cohen [4] presents a model similar to (1), where the carrying capacity is itself a function of the population. In a recent paper, Cai [3] has generalized this approach by considering an economic growth model with carrying capacity described by a logistic equation in physical capital. He proved the resulting model to have saddle-node bifurcations. This paper is concerned with a generalization of the model analyzed in Cai [3]. Specifically, we assume that the capital accumulation equation contains a time delay which take cares of the previous occurring dynamics. Growth of capital at time $t$ is a function of the amount of capital held at time $t - \tau$, since newly-produced capital is available for use by time $\tau$. Notice that the idea of introducing a time delay into the dynamics of economic processes goes back to Kalecki [15], who first formally and systematically showed that a delay in production can cause cycles in the economy. This line of argument was later revived by the seminal article of Kydland and Prescott [16], who empirically analyzed how far time consuming investment, which they called time-to-build, could explain real business cycles. More recently, the issue of a time lag introduced in the spirit of Kalecki into models of growth theory has been extensively addressed in the literature (see, e.g., Asea and Zak [1]; Ferrara and Guerrini [6-7]; Guerrini [8-14]; Szydłowski [17-18]).

The present paper is organized as follows. After this introduction, Section 2 introduces the version model of Cai [3] with the generalization which includes the time delay. In Section 3, by choosing time delay as bifurcation parameter, we analyze the distribution of the roots of the characteristic equation, and find that there exists a stability switch as the delay increases. An Hopf bifurcation occurs when the delay passes through a sequence of critical values.

## 2 The delayed model formulation

As already mentioned in the introduction the mathematical model of Cai [3] is concerned with the modelling of an economy, where the population carrying capacity is a function of physical capital and grows according to the logistic law. Specifically, population is assumed to be determined by the fol-
following equation \( \dot{N} = \gamma N [1 - N/g(K)] \), where \( K \) denotes physical capital, \( g(K) = \bar{N}/[1 + (\bar{N}/N_0 - 1) e^{-cK}] \), \( N_0 \) is the initial population, \( N > N_0 \) is the final carrying capacity, and \( c > 0 \) is the growth coefficient of the carrying capacity. The economy consists of a single good \( Y \) that can be used either for consumption or investment. This good is produced by labor \( L \) and physical capital accordingly to a Cobb-Douglas production function \( Y = AK^\alpha L^{1-\alpha} \), where \( A \) is a positive constant that reflects the level of the technology and \( \alpha \in (0,1) \) is a constant. Each worker has a unit of time available each period that is supplied inelastically in the labor market. This allows us to identify the number of workers and the supply of labor each period. Additionally, we assume that there is full employment in the economy, so that employment and labor supply coincide, i.e. \( L = N \), and labor grows at a determined constant rate. Since the economy is closed, we have that the change in the capital stock equals gross investment less depreciation, i.e. \( \dot{K} = sAK^\alpha L^{1-\alpha} - \delta K \), where \( s \) denotes the constant saving rate and \( \delta \) is the constant depreciation rate of capital. In what follows we consider the delayed version of Cai's model [3], with a time lag introduced in the Kaleckian spirit. It is assumed that new capital is not produced instantaneously, but it is produced and installed after \( \tau \) periods. As a result, the model of economic growth with a built-in delay parameter in delivering capital goods is given by a system of two differential equations, one of which has a delayed argument:

\[
\begin{align*}
\dot{K} &= sAK^\alpha L^{1-\alpha} - \delta K_d, \\
\dot{L} &= \gamma L \left[ 1 - \frac{L}{g(K)} \right], \\
\end{align*}
\]

where \( K_d = K_{t-\tau} \). Equilibria (or steady states in the language of the economical sciences) of system (2), of course, coincide with the corresponding steady states for zero delay. They are obtained by setting the two equations of system (2) to zero and \( \tau = 0 \). Doing this, from Cai [3], we know there exist from one to three non-trivial equilibria \((K_*, L_*)\) such that \( sAK_*^{\alpha - 1} L_*^{1-\alpha} = \delta \) and \( g(K_*) = L_* \).

3 Stability and Hopf bifurcation

In this Section, by choosing the delay \( \tau \) as the bifurcation parameter and analyzing the associated characteristic equation of (2) at the positive equilibria, we investigate the local stability of equilibria and obtain the conditions under which system (2) undergoes Hopf bifurcations. Let \((K_*, L_*)\) be any arbitrary equilibrium of system (2) and let \( x = K - K_* \), \( y = L - L_* \). Then system (2)
becomes
\[
\begin{cases}
\dot{x} = sA (x_d + K_s) \alpha (y + L_s)^{1-\alpha} - \delta (x_d + K_s), \\
\dot{y} = \gamma (y + L_s) \left[1 - \frac{y + L_s}{g(x + K_s)}\right].
\end{cases}
\]  
(3)

The linearization of system (3) at the equilibrium (0, 0) is
\[
\begin{cases}
\dot{x} = (1 - \alpha) \delta K_s L_s^{-1} y, \\
\dot{y} = \gamma g'(K_s) x - \gamma y.
\end{cases}
\]  
(4)

The characteristic equation of (4) is given by
\[
\lambda^2 + \gamma \lambda + (\gamma + \lambda) ae^{-\lambda \tau} - a \gamma M = 0,
\]  
(5)

where
\[
a = (1 - \alpha) \delta > 0, \quad M = g'(K_s) K_s L_s^{-1} > 0.
\]

If there is no delay, Eq. (5) becomes
\[
\lambda^2 + (a + \gamma) \lambda + a \gamma (1 - M) = 0.
\]  
(6)

Since \(a + \gamma > 0\), the two roots of Eq. (6) are real and negative if \(M < 1\), and real with opposite signs if \(M > 1\). Moreover, if \(M = 1\), one root is zero. We can state the following Lemma.

**Lemma 3.1.** Let \(\tau = 0\). The positive equilibrium point \((K_s, L_s)\) of system (2) is asymptotically stable if \(M < 1\), it is unstable if \(M > 1\) and nonhyperbolic if \(M = 1\).

In what follows we study the existence of the Hopf bifurcation for system (4). We are looking for the values of \(\tau\) so that the equilibrium \((0, 0)\) changes from local asymptotic stability to instability or vice versa. It is well known that the trivial equilibrium of system (4) is locally asymptotically stable if each of its characteristic roots have negative real parts. Thus, the marginal stability is determined by the equations \(\lambda = 0\) and \(\lambda = i \omega\) \((\omega > 0)\). Substituting \(\lambda = 0\) into Eq. (5), one obtains that the characteristic equation (5) has a zero root if \(M = 1\). However, we will see that \(\lambda = 0\) is not a simple root. Hence, the system is a degenerated case and it is very difficult to determine the crossing direction of the characteristic roots through the imaginary axis. For simplicity, we will exclude this case in our analysis by assuming that \(M \neq 1\). Let \(\lambda = \pm i \omega\) represent the two purely imaginary roots of Eq. (5). Substituting \(i \omega\) \((\omega is a
positive real number) into Eq. (5), and separating the real and imaginary parts, we obtain the following two transcendental equations:

\[ \omega^2 + a\gamma M = a\gamma \cos \omega \tau + a\omega \sin \omega \tau, \]  
\[ \gamma \omega = a\gamma \sin \omega \tau - a\omega \cos \omega \tau. \]  

(7)  

(8)  

By squaring and adding (7) and (8), we find \( \omega \) to be solution of

\[ \omega^4 + \left( 2a\gamma M + \gamma^2 - a^2 \right) \omega^2 + a^2\gamma^2 \left( M^2 - 1 \right) = 0. \]  

(9)  

Setting \( p = 2a\gamma M + \gamma^2 - a^2 \) and \( q = a^2\gamma^2 \left( M^2 - 1 \right) \), Eq. (9) has discriminant \( \Delta = p^2 - 4q \). Note that \( q < 0 \) if \( M < 1 \) and \( q > 0 \) if \( M > 1 \). It is easy to see that if \( M < 1 \), Eq. (9) has exactly one positive root. If \( M > 1 \), \( p < 0 \) and \( p^2 > 4q \) (resp. \( p < 0 \) and \( p^2 = 4q \)), then Eq. (9) has two positive roots (resp. one positive root); if \( p \geq 0 \) or \( p < 0 \) and \( p^2 < 4q \), then it has no real roots or non-positive roots. Set

\[ B = -(M + \sqrt{M^2 - 1}) + \sqrt{(M + \sqrt{M^2 - 1})^2 + 1}. \]  

**Lemma 3.2.**

(1) Let \( M < 1 \). Then Eq. (9) has one positive root, say \( \omega_+ \).

(2) Let \( M > 1 \). Then Eq. (9) has two positive roots \( \omega_\pm \) if \( \gamma < aB \), one positive root \( \omega_+ \) if \( \gamma = aB \), and no positive root if \( \gamma \geq a(-1 + \sqrt{M^2 + 1}) \) or \( \gamma > aB \).

Without loss of generality, we suppose that Eq. (9) has two positive roots \( \pm i\omega_\pm \), with the possibility of \( \omega_+ = \omega_- \). Then, from Eqs. (7)-(8) we can determine

\[ \tau_j^\pm = \frac{1}{\omega_\pm} \arccos \left( \frac{\gamma^2 M}{\gamma^2 + \omega_\pm^2} \right) + \frac{2j\pi}{\omega_\pm}, \quad j = 0, 1, 2, ..., \]  

(10)  

at which Eq. (5) has a pair of purely imaginary roots \( \pm i\omega_\pm \). Denote by \( \lambda(\tau) = v(\tau) + i\omega(\tau) \) the root of Eq. (5) such that such that \( v(\tau_j^\pm) = 0 \) and \( \omega(\tau_j^\pm) = \omega_\pm \). Substituting \( \lambda(\tau) \) into (5) and taking the derivative with respect to \( \tau \) yields

\[ \left[ 2\lambda + \gamma + ae^{-\lambda \tau} - (\gamma + \lambda) \tau ae^{-\lambda \tau} \right] \frac{d\lambda}{d\tau} = (\gamma + \lambda) \lambda ae^{-\lambda \tau}. \]  

(11)  

From (11) it is easily seen that \( \lambda = 0 \) is not a simple characteristic root of (5), whether \( \lambda = i\omega_\pm \) are. For example, if \( \lambda = i\omega_\pm \) is a repeated root, then
from Eq. (11) we have \((\gamma + i\omega_\pm) ia\omega_\pm = 0\), which is a contradiction. Eq. (5) together with (11) leads to

\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda + \gamma + ae^{-\lambda\tau}}{(\gamma + \lambda) \lambda ae^{-\lambda\tau}} - \frac{\tau}{\lambda} = \frac{(\gamma + \lambda)^2 + a\gamma M}{(\gamma + \lambda)(a\gamma M - \lambda^2 - \gamma\lambda)} - \frac{\tau}{\lambda}.
\]

Substituting (9) into it, we find

\[
\text{sign} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \bigg|_{\tau=\tau_j^\pm} \right\} = \text{sign} \left\{ \text{Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \bigg|_{\tau=\tau_j^\pm} \right\} = \text{sign} \left\{ \frac{(\gamma^2 - a\gamma M - \omega_\pm^2)^2 + 2\gamma^2 (a\gamma M + 2\omega_\pm^2)}{\gamma^2 (a\gamma M + 2\omega_\pm^2)^2 + (-a\gamma M + \gamma^2 - \omega_\pm^2)^2 \omega_\pm^2} \right\} = \text{sign} \left\{ \frac{\gamma^2 - a\gamma M - \omega_\pm^2}{\gamma^2 (a\gamma M + 2\omega_\pm^2)} \right\} = 1.
\]

Hence,

\[
\text{Re} \left[ \lambda'(\tau_j^\pm) \right] \equiv \left. \frac{d[\text{Re}\lambda(\tau)]}{d\tau} \right|_{\tau=\tau_j^\pm} > 0.
\]

Since the sign is positive, each crossing of the real part of characteristic roots at \(\tau_j^\pm\) must be from left to right. Hence, the crossing direction is always toward instability. We can obtain the following results about the distribution of roots of Eq. (5).

**Theorem 3.3.** Let \(\tau_j^\pm (j = 0, 1, 2, \ldots)\) be defined as in (10).

1. Let \(M > 1\). If \(\gamma \geq a(-1 + \sqrt{M^2 + 1})\) or \(\gamma > aB\) is satisfied, then all roots of Eq. (5) have negative real parts for all \(\tau \geq 0\).

2. Let \(M < 1\) or \(M > 1\) and \(\gamma = aB\). Then Eq. (5) has a pair of simple imaginary roots \(\pm i\omega_+\) at \(\tau = \tau_j^+, j = 0, 1, 2, \ldots\). Furthermore, if \(\tau \in [0, \tau_0^+),\) then all roots of Eq. (5) have negative real parts. If \(\tau = \tau_0^+\), then all roots of (5) except \(\pm i\omega_+\) have negative real parts. If \(\tau \in (\tau_j^+, \tau_{j+1}^+)\) for \(j = 0, 1, 2, \ldots,\) Eq. (5) has \(2(j + 1)\) roots with positive real parts.

3. Let \(M > 1\) and \(< \gamma < aB\). Then Eq. (5) then has a pair of simple purely imaginary roots \(\pm i\omega_+\) (\(\pm i\omega_-\), respectively) at \(\tau = \tau_j^+ (\tau = \tau_j^-,\) respectively), \(j = 0, 1, 2, \ldots\). If \(\tau \in [0, \tau_0),\) where \(\tau_0 = \min\{\tau_0^+, \tau_{0}^+\},\) then all roots of Eq. (5) have negative real parts. If \(\tau = \tau_0,\) then all roots of (5) except \(\pm i\omega_\pm\) have negative real parts. If \(\tau > \tau_0,\) Eq. (5) has roots with positive real parts.
According to the above analysis, one has the following result on the stability and Hopf bifurcation of system (2).

**Theorem 3.4.**

1. Let $M > 1$. When $\gamma \geq a(-1 + \sqrt{M^2 + 1})$ or $\gamma > aB$, the positive equilibrium $(K_*, L_*)$ of system (2) is asymptotically stable for $\tau \geq 0$.

2. Let $M < 1$ or $M > 1$ and $\gamma = aB$. The positive equilibrium $(K_*, L_*)$ of system (2) is asymptotically stable for $\tau \in [0, \tau_0^+)$ and unstable for $\tau \in (\tau_j^+, \tau_{j+1}^+)$ for $j = 0, 1, 2, \ldots$. Hopf bifurcation occurs when $\tau = \tau_j^+$, $j = 0, 1, 2, \ldots$.

3. Let $M > 1$ and $\gamma < aB$. The positive equilibrium $(K_*, L_*)$ of system (2) is asymptotically stable for $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{\tau_0^-, \tau_0^+, \tau_j^+, \tau_{j+1}^+\}$, and unstable for $\tau > \tau_0$. Hopf bifurcation occurs when $\tau = \tau_j^\pm$, $j = 0, 1, 2, \ldots$.

**References**


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