Approximate Fixed Point of Uniform Hemicontactive Mapping and to Apply to Iterative Solution of Equation with Uniform-accretive Mapping

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Abstract

The objective of this paper is to introduce the uniform-hemicontactive mapping and to study iterative approximative method for the fixed point of the mappings by Mann iterative sequence with random errors. Let $X$ be a real Banach space and $T : X \to X$ an uniform-hemicontactive the results presented in this paper show that Mann iterative sequence with random errors converges strongly to an unique fixed point if $T$ is uniformly continuous. Furthermore, if $X$ is uniformly smooth then any continuity of $T$ is unnecessary for the convergence of Mann iterative sequence. As applications, using these results, the iterative solution of nonlinear equation with uniform-accretive mapping is obtained.

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1 Introduction and Main Conclusions

Throughout this paper, $X$ is assumed a real Banach space with dual $X^*$, $(\cdot, \cdot)$ denotes the generalized duality pairing of $X$ and $X^*$. The mapping
$J : X \to 2^{X^*}$ defined by

$$Jx = \{ j \in X^* : (x, j) = \|x\|\|j\|, \|j\| = \|x\| \} \quad \forall x \in X$$

(1)

is called the normalized duality mapping. In particular, $X$ is an uniformly smooth (equivalently, $X^*$ is uniformly convex) Banach space if and only if $J$ is single-valued and uniformly continuous on any bounded subset of $X$(see, Browder[2]).

The $\phi$-hemicontractive mapping was introduced and studied by Osilike[6] in 1996. He proved that both the Mann[14] and Ishikawa[12] iterative sequences converge strongly to the unique fixed point of $T$ under certain conditions where $T$ is Lipschitz $\phi$-hemicontractive. Recently, Z. Q. Liu[18] proved that if $X$ is an arbitrary real Banach space and $T : X \to X$ is an uniformly continuous $\phi$-hemicontractive mapping, the Mann or Ishikawa iteration sequences with random errors introduced by Y. Xu[17] converge strongly to the fixed point of $T$ under certain conditions.

The objective of this paper is to introduce the uniform-hemicontractive mappings – a class of mappings which is much more general than the important class of $\phi$-hemicontractive mappings, and to study iterative approximation method for the fixed point of uniform-hemicontractive mapping and the solution of equation with uniform-accretive mapping by Mann iterative process with random errors defined by Definition 1.2 below. The results presented in this paper show that Mann iteration sequence $\{x_n\}$ converges strongly to an unique fixed point of $T$ if it is uniformly continuous. Furthermore, if $X$ is uniformly smooth then any continuity of $T$ is unnecessary for the convergence of $\{x_n\}$. As application, using these results, the iterative solution of nonlinear equation with uniform-accretive mapping is obtained in the more general settings.

To set the framework, we recall some basic notations as follows.

**Definition 1.1** Let $T$ or $A$ be mappings with domain $D(T) \subset X$ (or $D(A) \subset X$) and range $R(T) \subset X$ (or $R(A) \subset X$).

a) $T$ is called $\phi$-hemicontractive if for all $x \in D(T)$ and $q \in F(T) := \{ x \in D(T) : Tx = x \}$ there exist $j(x - q) \in J(x - q)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$(Tx - q, j(x - q)) \leq \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|. \quad (2)$$

b) $T$ is called uniform-hemicontractive if for all $x \in D(T)$ and $q \in F(T)$ there exist $j(x - q) \in J(x - q)$ and a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$(Tx - q, j(x - q)) \leq \|x - q\|^2 - \Phi(\|x - q\|). \quad (3)$$
c) A is called uniform-accretive if for all \( x, y \in D(A) \) there exist \( j(x - y) \in J(x - y) \) and a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \) such that
\[
(Ax - Ay, j(x - y)) \geq \Phi(\|x - y\|). \tag{4}
\]
d) A is called accretive if for all \( x, y \in D(A) \) there exist \( j(x - y) \in J(x - y) \) such that
\[
(Ax - Ay, j(x - y)) \geq 0.
\]

Obvious, if \( q \) is a fixed point of uniform-hemicontactive mapping \( T \), then \( q \) is unique. And every \( \phi \)-hemicontactive mapping must be an uniform-hemicontactive mapping defined by \( \Phi(s) = \phi(s)s \). The following example shows that the class of \( \phi \)-hemicontactive mappings is a proper subset of the class of uniform-hemicontactive mappings.

**Example.** Let \( E = \mathbb{R} \) (the reals with the usual norm) and let \( K = [0, +\infty) \). Define \( T : K \to K \) by
\[
Tx = x - \frac{x}{1 + x^2}.
\]
It is easy to verify that \( T \) is uniform-hemicontactive with a fixed point \( x = 0 \) and \( \Phi : [0, +\infty) \to [0, +\infty) \) defined by \( \Phi(s) = s^2/(1 + s^2) \). However, \( T \) is not \( \phi \)-hemicontactive. In fact, if there exists a strictly increasing function \( \phi^* : [0, \infty) \to [0, \infty) \) with \( \phi^*(0) = 0 \) satisfying (2), we get an inequality \( \phi^*(x) \leq x/(1 + x^2) \) for all \( x \in (0, +\infty) \), and so \( \lim_{x \to +\infty} \phi^*(x) \leq \lim_{x \to +\infty} x/(1 + x^2) = 0 \). This is incompatible with the strictly monotonicity of \( \phi^* \) on \([0, \infty)\) and \( \phi^*(0) = 0 \).

Let \( K \) be a nonempty convex subset of \( X \), and \( T : K \to K \) be a mapping. For any given \( x_0 \in K \) the sequence \( \{x_n\} \) defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n \quad (n \geq 0)
\]
is called Mann iteration sequence, where \( \{\alpha_n\} \) is a sequences in \([0, 1]\) satisfying some conditions. The consideration of error terms is an important part of any theory of iteration methods, for this reason, we introduced the following definition.

**Definition 1.2** Let \( T : X \to X \) be a mapping. For any given \( x_0 \in X \) the sequence \( \{x_n\} \) defined by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n + \gamma_nu_n \quad (n \geq 0)
\]
is called Mann iteration sequence with random errors. Here \( \{u_n\} \) is a bounded sequence in \( X \); \( \{\alpha_n\} \), and \( \{\gamma_n\} \) are sequences in \([0, 1]\).

The following Lemma play crucial role in the proofs of our main results.
Lemma 1.3 \cite{1,10}. If $X$ be a real Banach space then there exists $j(x+y) \in J(x+y)$ such that
\[
\|x+y\|^2 \leq \|x\|^2 + 2(y, j(x+y)) \quad \forall x, y \in X. \tag{6}
\]

Now we provide approximative theorems as follows.

**Theorem 1.4** Suppose that $T : X \to X$ is an uniformly continuous and uniform-hemicontractive mapping with bounded range. If the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by (5) satisfying
\[
\lim_{n \to \infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{+\infty} \alpha_n = +\infty;
\]
then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique fixed point of $T$.

**Theorem 1.5** Let $T : X \to X$ be an uniform-hemicontractive mapping with bounded range and $X$ be uniformly smooth. Suppose that the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by (5) satisfying the conditions (1.4.1) and (1.4.2) in Theorem 1.4, then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique fixed point of $T$.

The accretive mappings were introduced independently in 1967 by Browder\cite{3} and Kato\cite{13}. An early fundamental result in the theory of accretive mappings, due to Browder, states that the initial value problem $dx/dt + Ax = 0, \quad x(0) = x_0$ is solvable if $A$ is locally Lipschitzian and accretive on $X$. Martin\cite{9} generalized the result of Browder to the continuous strongly accretive mapping. That is, he proved that if $A : X \to X$ is strongly accretive and continuous, then $A$ is surjective, so that the equation
\[
Ax = f \tag{7}
\]
has a solution for any given $f \in X$. On the other hand, he established also that if $A : X \to X$ is accretive and continuous, then the equation $x + Ax = f$ has a solution for any given $f \in X$.

**Remark 1.1.** Suppose that $A : X \to X$ is an uniform-accretive mapping and $S : X \to X$ is defined by $Sx = f + x - Ax$ for all $x \in X$ and any given $f \in X$, it is easy to verify that $q$ is a solution of Eq. (7) if and only if $q$ is a fixed point of $S$. Hence, the solution of Eq. (7) is intimately connected with the fixed point of the mapping.

**Corollary 1.6** Suppose that $A : X \to X$ is an uniformly continuous and uniform-accretive mapping and the range of $(I - A)$ is bounded. If the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by
\[
x_0 \in X, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sx_n + \gamma_n u_n \quad (n \geq 0)
\]
satisfying the conditions (1.4.1) and (1.4.2) in Theorem 1.4, where $S : X \to X$ defined by $Sx = f + x - Ax$, and for any given $f \in X$ the equation $Ax = f$ has a solution, then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique solution of $Ax = f$.

Similarly, we have

Corollary 1.7 Let $A : X \to X$ be an uniform-accretive mapping, the range of $(I - A)$ be bounded and $X$ be uniformly smooth. Suppose that the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in X, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + \gamma_n u_n \quad (n \geq 0)$$

satisfying the conditions (1.4.1) and (1.4.2) in Theorem 1.4, where $S : X \to X$ defined by $Sx = f + x - Ax$, and for any given $f \in X$ the equation $Ax = f$ has a solution, then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the unique solution of $Ax = f$.

As a consequence of Corollary 1.6, we obtain

Corollary 1.8 Suppose that $A : X \to X$ is an uniformly continuous and uniform-accretive mapping with bounded range. If the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in X, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + \gamma_n u_n \quad (n \geq 0)$$

satisfying the conditions (1.4.1) and (1.4.2) in Theorem 1.4, where $S : X \to X$ defined by $Sx = f - Ax$, then for arbitrary $x_0 \in X$ and for any given $f \in X$, $\{x_n\}$ converges strongly to the unique solution of $x + Ax = f$.

Similarly, we also have

Corollary 1.9 Let $A : X \to X$ be an uniform-accretive mapping with bounded range and $X$ be uniformly smooth. If the Mann iteration sequence with random errors $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in X, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + \gamma_n u_n \quad (n \geq 0)$$

satisfying the conditions (1.4.1) and (1.4.2) in Theorem 1.4, where $S : X \to X$ defined by $Sx = f - Ax$, then for arbitrary $x_0 \in X$ and for any given $f \in X$, $\{x_n\}$ converges strongly to the unique solution of $x + Ax = f$. 
Remark 1.2. Theorem 1.4, Theorem 1.5 and Corollary 1.6—Corollary 1.9 have been shown that we can approximate to the fixed point of uniform-hemicontractive mapping or the solution of equations with uniform-accretive mapping by the Mann iteration sequence with random errors. The corresponding results (see, for example, Theorem 4.1 and Theorem 4.2 of [11], Theorem 3 and Theorem 4 of [8], Corollary 2.4 of [7], Corollary 3.3 of [18], Corollary 3.2 and Corollary 3.4 of [17], Theorem 2 of [4], Corollary 3.2 of [15] and Corollary 3.2 of [16]) are improved in the following senses:

i) for the convergence of the mann iterative sequence with random errors, if \( X \) is arbitrary Banach space, the mapping may not be Lipschitz, if \( X \) is an uniformly smooth Banach space, the mapping may not be continuous or demi-continuous, therefore, it may not be Lipschitz, also;

ii) the mapping may not be \( \phi \)-hemicontractive or \( \phi \)-strongly accretive;

iii) the random errors of iterative process have been considered appropriately again.

Remark 1.3. The Mann iteration process with errors on a nonempty convex subset of Banach space was introduced in 1995 by L.S. Liu [5] first. Xu, one of authors, revised the definition of Liu in 1998 (see, Xu [17]). Definition 1.2 differs from the definitions in [5] and [17]. Note that the Mann iterative process is a special case of the Mann iteration process with random errors. By the way, the iterative parameters \( \{\alpha_n\} \) and \( \{\gamma_n\} \) do not depend on any geometric structure of space \( X \) and on any property of the mappings, but, the selection of the parameters is deal with the convergence rate of the iterative sequence. In this paper, a prototype of iteration parameters is

\[
\alpha_n = \frac{1}{n+1} \quad \text{and} \quad \gamma_n = \frac{1}{n!} \quad \forall \quad n \geq 0.
\]

2 Proofs of Main Conclusions

Now we prove the approximative theorems.

The proof of Theorem 1.4.

Proof. We know that \( F(T) = \{q\} \).

Putting

\[
\begin{align*}
c &= \sup \{\|Tx - q\| : x \in X\} + \|x_0 - q\| \\
d &= \sup \{\|u_n\| : n \geq 0\}.
\end{align*}
\]

For any \( n \geq 0 \), using induction, we obtain

\[
\|x_n - q\| \leq c + d \sum_{i=0}^{n-1} \gamma_i \leq c + d \sum_{i=0}^{+\infty} \gamma_i.
\]

Hence, we set

\[
M = c + d \sum_{i=0}^{+\infty} \gamma_i.
\]
Since
\[ \lim_{n \to \infty} \|x_n - x_{n+1}\| = \lim_{n \to \infty} \|\alpha_n x_n - \alpha_n T x_n - \gamma_n u_n\| = 0, \]
therefore,
\[ e_n := \|T x_n - T x_{n+1}\| \to 0 \quad (as \ n \to \infty) \]
by the uniformly continuity of \(T\).

Let \(\sigma = \inf\{\|x_{n+1} - q\| : n \geq 0\}\). If \(\sigma > 0\) then \(\Phi(\|x_{n+1} - q\|) > \Phi(\sigma/2) > 0\)
for all \(n \geq 0\). Thus, there exists a natural number \(N \in \mathbb{N}\) such that
\[ \alpha_n \leq \frac{1}{6} \quad \text{and} \quad M^2 \alpha_n + 2M e_n \leq \Phi\left(\frac{\sigma}{2}\right) \quad \forall \ n \geq N, \]
respectively. By (3), (6) and (9), we have
\[
\begin{align*}
\|x_{n+1} - q\|^2 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n (T x_n - q) + \gamma_n u_n\|^2 \\
&\leq \|(1 - \alpha_n)(x_n - q)\|^2 \\
&\quad + 2 \alpha_n (T x_n - q) + \gamma_n u_n, \ j(x_{n+1} - q) \\
&\leq \|(1 - \alpha_n)(x_n - q)\|^2 + 2 \alpha_n (T x_n - T x_{n+1}, \ j(x_{n+1} - q) \\
&\quad + 2 \alpha_n (T x_{n+1} - q, \ j(x_{n+1} - q)) + 2M^2 \gamma_n \\
&\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2M \alpha_n e_n + 2 \alpha_n \|x_{n+1} - q\|^2 \\
&\quad - 2 \alpha_n \Phi(\|x_{n+1} - q\|) + 2M^2 \gamma_n \\
&\leq (1 - 2 \alpha_n) \|x_n - q\|^2 + 2 \alpha_n \|x_{n+1} - q\|^2 \\
&\quad + \alpha_n [M^2 \alpha_n + 2M e_n - \Phi(\sigma/2)] + 2M^2 \gamma_n - \Phi(\sigma/2) \alpha_n \\
&\leq \|x_n - q\|^2 + \frac{2}{1 - 2 \alpha_n} M^2 \gamma_n - \Phi(\sigma/2) \alpha_n \\
&\leq \|x_n - q\|^2 + 3M^2 \gamma_n - \Phi(\sigma/2) \alpha_n
\end{align*}
\]
for all \(n \geq N\).

By induction, we have
\[ 0 \leq \|x_N - q\|^2 + 3M^2 \sum_{j=N}^{+\infty} \gamma_j - \Phi\left(\frac{\sigma}{2}\right) \sum_{j=N}^{+\infty} \alpha_j. \]
I.e.,
\[ \Phi\left(\frac{\sigma}{2}\right) \sum_{j=N}^{+\infty} \alpha_j \leq \|x_N - q\|^2 + 3M^2 \sum_{j=N}^{+\infty} \gamma_j < +\infty. \tag{11} \]

(11) is in contradiction with \(\sum_{j=0}^{+\infty} \alpha_j = +\infty\). From this contradiction, we know
that \(\sigma = 0\). Therefore, there exists a subsequence \(\{x_{n_j+1}\} \subset \{x_{n+1}\}\) such that
\(x_{n_j+1} \to q\) (as \(n_j \to \infty\)). By induction, we can prove
\[ x_{n_j+k} \to q \quad (as \ n_j \to \infty) \quad \forall k > 0. \]
This implies that \(x_n \to q\). The Proof is completed.
The proof of Theorem 1.5.

**Proof.** We know that the fixed point of \( T \) is unique. Let \( q \) be the fixed point of \( T \) in \( X \). By similar arguments as in the proof of Theorem 1.4, we set

\[
M = c + d \sum_{i=0}^{+\infty} \gamma_i.
\]

Using (8) and the uniformly continuity of \( J \), we have

\[
e_n := \|J(x_{n+1} - q) - J(x_n - q)\| \to 0 \text{ (as } n \to \infty)\]

From (3), (6) and (9), we have

\[
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(x_n - q) + \alpha_n(Tx_n - q) + \gamma_n u_n\|^2 \\
\leq \|(1 - \alpha_n)(x_n - q)\|^2 + 2\alpha_n(Tx_n - q, J(x_{n+1} - q)) \\
+ 2\gamma_n(u_n, J(x_{n+1} - q)) \\
\leq \|(1 - \alpha_n)(x_n - q)\|^2 + 2\alpha_n(Tx_n - q, J(x_n - q)) \\
+ 2\alpha_n(Tx_n - q, J(x_{n+1} - q) - J(x_n - q)) + 2M^2\gamma_n \\
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\|x_n - q\|^2 \\
- 2\alpha_n\Phi(\|x_n - q\|) + 2M\alpha_n e_n + 2M^2\gamma_n \\
\leq (1 + \alpha_n^2)\|x_n - q\|^2 - 2\alpha_n\Phi(\sigma/2) + 2M\alpha_n e_n + 2M^2\gamma_n \\
\leq \|x_n - q\|^2 + 2M^2\gamma_n - \Phi(\sigma/2)\alpha_n \\
+ \alpha_n[M^2\alpha_n + 2Me_n - \Phi(\sigma/2)] \\
\leq \|x_n - q\|^2 + 2M^2\gamma_n - \Phi(\sigma/2)\alpha_n.
\]

(12)

By similar arguments as in the proof of Theorem 1.4, we have that \( \{x_n\} \) converges strongly to the unique fixed point \( q \) of \( T \). The Proof is completed.

Now, we prove corollaries as follows.

The proof of Corollary 1.6.

**Proof.** Putting \( S : X \to X \) by \( Sx = f + x - Ax \) for all \( x \in X \). Obvious, if \( q \in X \) is a solution of Eq. (7) then \( q \) is a fixed point of \( S \) and \( S \) is uniform-hemicontractive. Thus Corollary 1.6 follows from Theorem 1.4.

The proof of Corollary 1.7.

**Proof.** Similarly, the conclusion from Theorem 1.5.

The proof of Corollary 1.8.

**Proof.** Since \( A \) is accretive and continuous then \( A \) is \( m \)-accretive, so that the equation \( x + Ax = f \) has a solution \( x = q \) for any given \( f \in X \). Putting \( A_0 = I + A \), the equation \( x + Ax = f \) becomes \( A_0x = f \). It is easy to see that \( A_0 \) is uniformly continuous and uniform-accretive. Thus Corollary 1.8 follows from Corollary 1.6.
The proof of Corollary 1.9.

Proof. In fact, Corollary 1.9 is a direct result of Corollary 1.7.

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