Extending Property (T) to $C^*$-algebras

Firuz Kamalov

Mathematics Department, Canadian University of Dubai
Dubai, UAE
firuz@cud.ac.ae

Copyright © 2013 Firuz Kamalov. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this note we compare a pair of alternative definitions of Property (T) in the context of $C^*$-algebras. The first definition was introduced by Bekka and the second by Pavlov and Troitsky. Although the two definitions are equivalent for group $C^*$-algebras they are different in general. We show that the definition of Bekka is in a certain sense stronger than the definition of Pavlov and Troitsky. In addition, we analyze both definitions in the case of abelian $C^*$-algebras.

Mathematics Subject Classification: 46L55, 46L05

Keywords: Property (T), $C^*$-algebras

1 Introduction

Property (T) was originally introduced by Kazhdan to study the lattice structure of groups [7]. It has been since studied in many other mathematical contexts including ergodic theory, dynamical systems, and graph theory [2]. Our goal in this paper is to investigate recent attempts to extend Property (T) to $C^*$-algebras. Let $G$ be a locally compact group. There exist two common ways of defining Property (T) for $G$.

Definition 1.1. Let $G$ be a locally compact group. Then $G$ has Property (T) if it satisfies one of the following equivalent conditions:

1. There exists a compact subset $S \subseteq G$ and $\epsilon > 0$ such that the following property holds: if a unitary representation $\pi$ of $G$ contains a $(S, \epsilon)$-central
unit vector, then it contains a non-zero central vector, that is, a vector \( \xi \in \mathcal{H}_\pi \) such that \( \pi(s)\xi = \xi \) for all \( s \in G \).

2. The trivial representation \( 1_G \) is isolated in the unitary dual \( \hat{G} \) of \( G \).

Starting with the work of Connes [4] various authors tried to define Property (T) for \( C^* \)-algebras [1, 6, 8]. There are two natural ways to extend the definition of Property (T) from groups to \( C^* \)-algebras. The first approach was used by Bekka and the second approach was used by Pavlov and Troitsky. To distinguish between the two notions of Property (T) for \( C^* \)-algebras we will denote the definitions of Bekka and Pavlov by \( (T_B) \) and \( (T_P) \) respectively.

Recall that a Hilbert bimodule over a \( C^* \)-algebra \( A \) is a Hilbert space \( \mathcal{H} \) carrying a pair of commuting representations, one of \( A \) and one of its opposite algebra \( A^o \). Bekka’s definition is in the spirit of Condition (1) in Definition 1.1.

**Definition 1.2.** Let \( A \) be a unital \( C^* \)-algebra. Then \( A \) has Property \( (T_B) \) if there is a finite subset \( F \subseteq A \) and \( \epsilon > 0 \) such that the following property holds: if a Hilbert bimodule \( \mathcal{H}_\pi \) of \( A \) contains a unit vector \( \xi \) which is \((F, \epsilon)\)-central, that is, such that \( \| \pi(x)\xi - \pi^o(y)\xi \| < \epsilon \) for all \( x, y \in F \), then \( \mathcal{H}_\pi \) has a non-zero central vector, that is, a vector \( \zeta \in \mathcal{H}_\pi \) such that \( \pi(x)\zeta = \pi^o(y)\zeta \) for all \( x, y \in A \).

Bekka showed that \( G \) has Property (T) if and only if its reduced group \( C^* \)-algebra \( C^*_r(G) \) has Property \( (T_B) \) [1]. In addition, several important facts regarding Property (T) for groups were extended to \( C^* \)-algebras using Bekka’s definition [3, 6].

Pavlov and Troitsky proposed an alternative definition of Property (T) for \( C^* \)-algebras based on Condition (2) in Definition 1.1. In particular, they showed that the Property \( (T_P) \) is equivalent to property DINC [8].

**Definition 1.3.** Let \( A \) be a unital \( C^* \)-algebra. Then \( A \) has Property \( (T_P) \) if it has a finite dimensional irreducible representation \( \pi \) such that \( \pi \) is isolated in the dual \( \hat{A} \) of \( A \).

Although Property \( (T_B) \) and \( (T_P) \) are equivalent when \( A = C^*_r(G) \) they are different in general. In particular, if \( A \) does not have any tracial states, then \( A \) has Property \( (T_B) \), but it does not have Property \( (T_P) \). If \( A \) does have a tracial state, then the situation is less clear. In Section 1 we show that Property \( (T_B) \) implies Property \( (T_P) \) when \( A \) has a finite dimensional representation. In Section 2 we compare the two properties in the context of abelian \( C^* \)-algebras.
2 Property \((T_B)\) implies Property \((T_P)\)

Let \(A\) be a unital \(C^*\)-algebra. Recall that a tracial state on \(A\) is a positive linear functional \(\phi: A \to \mathbb{C}\) such that \(\phi(xy) = \phi(yx)\) for all \(x, y \in A\) and \(\phi(1_A) = 1\). If \(A\) does not admit any tracial states the questions regarding Property \((T)\) can be answered definitively.

**Proposition 2.1.** Let \(A\) be a unital \(C^*\)-algebra. Suppose that \(A\) does not admit any tracial states. Then the following statements are true:

1. \(A\) satisfies Property \((T_B)\);

2. \(A\) does not satisfy Property \((T_P)\).

**Proof.** The first statement was shown by Bekka in [1, Remark 17]. To prove the second statement, let \(\pi\) be an \(n\)-dimensional representation of \(A\). Let \(tr\) denote the canonical trace on \(M_n(\mathbb{C})\). Then \(tr \circ \pi\) is a tracial state on \(A\). By hypothesis, \(A\) does not have any tracial states. It follows that \(A\) does not satisfy Property \((T_P)\). \(\square\)

If \(A\) admits a tracial state the situation becomes more interesting. The next theorem shows that Property \((T_B)\) is in a certain sense stronger than Property \((T_P)\).

**Theorem 2.1.** Let \(A\) be a unital \(C^*\)-algebra. Suppose that \(A\) has a finite dimensional irreducible representation. If \(A\) satisfies Property \((T_B)\), then it satisfies Property \((T_P)\).

**Proof.** Let \(\pi\) be a finite dimensional irreducible representation of \(A\). Suppose that \(\pi\) is not isolated in \(\hat{A}\). Then there exists a net \(\{\pi_i\} in \hat{A}\) such that \(\pi_i \to \pi\) and \(\pi_i \neq \pi\). Let \(\rho = \oplus \pi_i\). Then \(\rho\) weakly contains \(\pi\). It follows by [3, Proposition 3.2] that \(\pi\) is contained in \(\rho\) which is a contradiction. \(\square\)

It is natural to ask if Property \((T_B)\) always implies Property \((T_P)\) whenever \(A\) has a tracial state. The answer seems to be negative, but we cannot construct an appropriate counter example. However, we have the following related example.

**Example 2.2.** Let \(\mathbb{T}\) be the unit circle and \(\mathbb{Z}\) be the group of integers. Let \(\theta\) be an irrational number in the unit interval \([0, 1]\). Define an action \(\sigma\) of \(\mathbb{Z}\) on \(C(\mathbb{T})\) by

\[\sigma_n(f)(z) = f(e^{-2\pi i n \theta} z).\]

The corresponding crossed product \(C^*\)-algebra \(A = C(\mathbb{T}) \times_\sigma \mathbb{Z}\) is called the "irrational rotation" algebra. Let \(\pi\) be the canonical representation of \(C(\mathbb{T})\)
on $L^2(\mathbb{T}, \mu)$, where $\mu$ is the Lebesgue measure. Let $\pi \times \lambda$ be, the induced, left regular representation of $A$ on $\mathcal{H} = L^2(G) \otimes L^2(\mathbb{T}, \mu)$. Define a tracial state on $A$ by
\[
\phi(x) = \langle \pi \times \lambda(x)(\chi_e \otimes 1_{\mathbb{T}}), \chi_e \otimes 1_{\mathbb{T}} \rangle,
\]
where $\chi_e$ is the characteristic function of the identity element in $G$ and $1_{\mathbb{T}}$ the constant function on the circle. It is well known that $A$ is a simple $C^*$-algebra. In particular, it has no finite dimensional irreducible representations and vacuously does not satisfy Property $(T_P)$. Moreover, $A$ is nuclear [5, Proposition 14] and hence it does not satisfy Property $(T_B)$ [1, Proposition 12].

3 Property (T) for $C(X)$

As was shown in Proposition 2.1, any $C^*$-algebra without a tracial state satisfies Property $(T_B)$, but does not satisfy Property $(T_P)$. In this section we will give an example of a $C^*$-algebra that satisfies Property $(T_P)$, but not Property $(T_B)$. Let $A$ be a unital abelian $C^*$-algebra $C(X)$, where $X$ is a second countable, compact Hausdorff space. We will show that if $C(X)$ has Property $(T_B)$, then $X$ must be finite. This result can also be obtained from [3, Theorem 5.1] or [1, Proposition 15], but we give an alternative proof. First we need the following lemma.

**Lemma 3.1.** Let $X$ be a second countable, compact Hausdorff space. Suppose that $X$ is infinite. Then there exists a finite Borel measure $\mu$ together with a point $x_0 \in X$ such that

1. $\mu(x_0) = 0$
2. $\mu(V) > 0$ for all open sets containing $x_0$.

**Proof.** First, suppose that $X$ is a countable set. Since $X$ is compact there must be an accumulation point $x_0 \in X$. It is easy to construct a finite measure $\mu$ on $X$ such that $\mu(x_0) = 0$ and $\mu(x) > 0$ for all $x \neq x_0$.

Suppose that $X$ is an uncountable set. By hypothesis, $X$ is completely metrizable. Therefore, $X$ is Borel isomorphic to the set $[0, 1]$ with its usual Borel structure. Let $\mu$ be the image of the Lebesgue measure on $X$ induced by the isomorphism. Clearly, $\mu(x) = 0$ for all $x \in X$. If for each $x \in X$ there is an open set $V$ containing $x$ such that $\mu(V) = 0$, then $\mu(X) = 0$ which is a contradiction. Therefore, there exists $x_0$ such that $\mu(V) > 0$ for all open sets $V$ containing $x_0$.

We are now in position to state the main result of this section.
Theorem 3.2. Let $X$ be a second countable, compact Hausdorff space. Suppose the $C^*$-algebra $C(X)$ has Property $(T_B)$. Then $X$ is a finite set.

Proof. Suppose that $X$ is not a finite set. Let $x_0 \in X$ and $\mu$ be as in Lemma 3.1. Define a representation $\pi$ of $C(X)$ on $L^2(X, \mu)$ by $\pi(f)(\xi) = f \cdot \xi$ for all $f \in C(X)$ and $\xi \in L^2(X, \mu)$. Similarly, define a representation $\rho$ of $C(X)$ on $L^2(X, \mu)$ by $\rho(f)(\xi) = f(x_0) \cdot \xi$ for all $f \in C(X)$ and $\xi \in L^2(X, \mu)$. Since $\pi(f)\rho(g) = \rho(g)\pi(f)$ for all $f, g \in C(X)$, then $L^2(X, \mu)$ is a Hilbert bimodule over $C(X)$. Since $X$ is a countable, compact Hausdorff space it is metrizable. Let $V_n$ be a sequence of open sets centered at $x_0$ with radius $\frac{1}{n}$. Define $\xi_n = \frac{1}{\mu(V_n)}\chi_{V_n}$. Then $\|\pi(f)\xi_n - \rho(f)\xi_n\| \to 0$ for all $f \in C(X)$. Since $C(X)$ satisfies property $(T_B)$, then there exists a non zero central vector in $L^2(X, \mu)$. In other words, there a unit vector $\xi_0 \in L^2(X, \mu)$ such that $f \cdot \xi_0 = f(x_0) \cdot \xi_0$ for all $f \in C(X)$. This implies that $\mu(x_0) = 1$ which is a contradiction. Hence, $X$ is a finite set.

Define $X = [0, 1] \cup \{2\}$. Then the $C^*$-algebra $C(X)$ satisfies Property $(T_P)$, but it does not satisfy Property $(T_B)$ by the above theorem.

References


Received: January 17, 2013