Subclass of Multivalent Functions Defined by Hadamard Product Involving a Linear Operator

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Abstract
In the present paper, we introduce a subclass $R_p(γ, β, m, A, B, α)$ of multivalent analytic functions in the open unit disc $U$. We study coefficient inequalities, closure theorem, neighborhood property and partial sums, radii of starlikeness, convexity and close-to-convexity. We also obtain weighted mean, arithmetic mean and linear combination.

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1 Introduction

Let $A_p$ denote the class of all functions of the form:
\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in N = \{1, 2, \cdots \}),
\]
which are analytic and multivalent in the open unit disk

\[ U = \{ z \in \mathbb{C} : |z| < 1 \}. \]

Let \( M_p \) denote the subclass of \( \mathcal{A}_p \) containing functions of the form:

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0, p \in \mathbb{N}),
\]

which are analytic and multivalent in the open unit disk \( U \).

For the functions \( f \in M_p \) given by (2) and \( g \in M_p \) defined by

\[
g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (b_k \geq 0, p \in \mathbb{N}).
\]

We define the convolution (or Hadamard product) of \( f \) and \( g \) by

\[(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.\]

A function \( f \in M_p \) is said to be \( p \)-valently starlike of order \( \rho \) if and only if

\[\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \rho, \quad (0 \leq \rho < p; z \in U).\]

A function \( f \in M_p \) is said to be \( p \)-valently convex of order \( \rho \) if and only if

\[\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \rho, \quad (0 \leq \rho < p; z \in U).\]

It follows from expression (5), (6) that \( f \) is convex if and only if \( zf'(z) \) is starlike. A function \( f \in M_p \) is close-to-convex of order \( \rho \) if

\[\Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \rho, \quad (0 \leq \rho < p; z \in U).\]

**Definition 1 [6]**: Let \( \gamma, \beta, m \in \mathbb{R}, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in \mathbb{N} \) and

\[f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.\]

Then, we define the linear operator

\[D_{p,m}^{\gamma,\beta} : \mathcal{A}_p \to \mathcal{A}_p \]

by
Subclass of multivalent functions

\[ D_{p,m}^{\gamma,\beta} f(z) = z^p + \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m a_k z^k, \quad z \in U. \]  

(8)

**Definition 2**: Let \( g \) be a fixed function defined by (3). The function \( f \in M_p \) given by (2) is said to be in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \) if and only if

\[
\frac{z(D_{p,m}^{\gamma,\beta}(f * g)(z))' - p(D_{p,m}^{\gamma,\beta}(f * g)(z))}{\lambda z(D_{p,m}^{\gamma,\beta}(f * g)(z))' + (A + B)(D_{p,m}^{\gamma,\beta}(f * g)(z))} < \alpha,
\]

where

\[ 0 < \lambda < 1, 0 < A < 1, 0 < B < 1, 0 < \alpha < 1, \gamma, \beta, m \in R, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in N. \]

Some of the following properties studied for other classes in [1], [2], [4] and [5].

### 2 Coefficient Inequalities

**Theorem 1**: Let \( f \in M_p \). Then \( f \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \) if and only if

\[
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_k b_k \leq \alpha[p\lambda + (A+B)],
\]

where

\[ 0 < \lambda < 1, 0 < A < 1, 0 < B < 1, 0 < \alpha < 1, \gamma, \beta, m \in R, \gamma \geq 0, \beta \geq 0, m \geq 0, p \in N. \]

The result is sharp for the function

\[ f(z) = z^p + \frac{\alpha[p\lambda + (A+B)]}{1 + \frac{(k-p)\gamma}{(p+\beta)}} [(k-p) - \alpha(\lambda k(A+B))] b_k z^k. \]

(11)

**Proof**: Suppose that the inequality (10) holds true and \(|z| = 1\). Then we have

\[
\begin{align*}
\frac{z(D_{p,m}^{\gamma,\beta}(f * g)(z))' - p(D_{p,m}^{\gamma,\beta}(f * g)(z))}{\lambda z(D_{p,m}^{\gamma,\beta}(f * g)(z))' + (A + B)(D_{p,m}^{\gamma,\beta}(f * g)(z))} & = |z(D_{p,m}^{\gamma,\beta}(f * g)(z))' - p(D_{p,m}^{\gamma,\beta}(f * g)(z))| \\
& \quad - \alpha |\lambda z(D_{p,m}^{\gamma,\beta}(f * g)(z))' + (A + B)(D_{p,m}^{\gamma,\beta}(f * g)(z))| \\
& = \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m (k-p) a_k b_k z^k.
\end{align*}
\]
Let $H$ be the given function. Hence, by maximum modulus principle, $f$ by hypothesis. Conversely, suppose that $f \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then from (9), we have

$$
\left| \frac{(D^\gamma_p f(z))^\prime - p(D^\gamma_p f(z))}{\lambda z(D^\gamma_p f(z))^\prime + (A + B)(D^\gamma_p f(z))} \right| \leq \sum_{k=p+1}^\infty \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m (k-p)a_k b_k |z|^k
$$

Since $Re(z) \leq |z|$ for all $z \in U$ we get

$$
Re \left\{ \sum_{k=p+1}^\infty \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m (k-p)a_k b_k |z|^k \right\} < \alpha.
$$

We choose the value of $z$ on the real axis so that $(D^\gamma_p f(z))^\prime$ is real. Letting $z \to 1^-$ through real values, we obtain inequality (10).

Finally, sharpness follows if we take

$$
f(z) = z^p + \frac{\alpha[p\lambda + (A + B)]}{1 + \frac{(k-p)\gamma}{(p+\beta)}} [(k-p) - \alpha(\lambda k + (A + B))]z^k,
$$

$k = p + 1, p + 2, \ldots$.

The proof is complete.

**Corollary 1**: Let $f \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$. Then

$$
a_k \leq \frac{\alpha[p\lambda + (A + B)]}{1 + \frac{(k-p)\gamma}{(p+\beta)}} [(k-p) - \alpha(\lambda k + (A + B))]z^k, \quad k = p+1, p+2, \ldots
$$
3 Closure Theorem

**Theorem 2**: Let the functions $f_s$ defined by

$$f_s(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,s} z^k, \quad (a_{k,s} \geq 0, p \in N, s = 1, 2, \cdots, l),$$

be in the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ for every $s = 1, 2, \cdots, l$. Then the function $h$ defined by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} e_k z^k, \quad (e_k \geq 0, p \in N),$$

also belongs to the class $R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, where

$$e_k = \frac{1}{l} \sum_{s=1}^{l} a_{k,s}, \quad (k \geq p + 1).$$

**Proof**: Since $f_s \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, then by Theorem 1, we have

$$\sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_{k,s} b_k \leq \alpha[p\lambda + (A+B)], \quad (15)$$

for every $s = 1, 2, \cdots, l$. Hence

$$\sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m [(k-p) - \alpha(\lambda k + (A+B))] e_k b_k$$

$$= \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m [(k-p) - \alpha(\lambda k + (A+B))] b_k \left( \frac{1}{l} \sum_{s=1}^{l} a_{k,s} \right)$$

$$= \frac{1}{l} \sum_{s=1}^{l} \left( \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m [(k-p) - \alpha(\lambda k + (A+B))] a_{k,s} b_k \right)$$

$$\leq \alpha[p\lambda + (A+B)].$$

By Theorem 1, it follows that $h \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$.

4 Neighborhoods and Partial Sums

We define the $(n, \delta)$-neighborhood of a function $f \in M_p$ by

$$N_{n,\delta}(f) = \left\{ g \in M_p : g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \text{ and } \sum_{k=p+1}^{\infty} k|a_k - b_k| \leq \delta, \quad 0 \leq \delta < 1 \right\}. \quad (16)$$
For the identity function $e(z) = z$, $(p \in N)$

$$N_{n,\delta}(e) = \left\{ g \in M_p : g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=p+1}^{\infty} k|b_k| \leq \delta, \quad 0 \leq \delta < 1 \right\}.$$  

(17)

The concept of neighborhoods was first introduced by Goodman [3] and then generalized by Ruscheweyh [7].

**Definition 3**: A function $f \in M_p$ is said to be in the class $R^n(\gamma, \beta, m, \lambda, A, B, \alpha)$ if there exists a function $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \eta \quad (z \in U, \quad 0 \leq \eta < 1).$$

**Theorem 3**: If $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$ and

$$\eta = p - \frac{\delta \left(1 + \frac{\gamma}{p+\beta}\right)^m \left[1 - \alpha(p+1) + (A + B)\right]a_{p+1}}{(p+1) \left(1 + \frac{\gamma}{p+\beta}\right)^m \left[1 - \alpha(p+1) + (A + B)\right]a_{p+1} - \alpha[p\lambda + (A + B)]}.$$  

(18)

Then $N_{n,\delta}(g) \subset R^n(\gamma, \beta, m, \lambda, A, B, \alpha)$.

**Proof**: Let $f \in N_{n,\delta}(g)$. We want to find from (16) that

$$\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \delta,$$

which readily implies the following coefficient inequality

$$\sum_{k=p+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{p+1}.$$  

Next, since $g \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha)$, we have from Theorem 1

$$\sum_{k=p+1}^{\infty} b_k \leq \frac{\alpha[p\lambda + (A + B)]}{\left(1 + \frac{\gamma}{p+\beta}\right)^m \left[1 - \alpha(p+1) + (A + B)\right]a_{p+1}}.$$  

So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=p+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=p+1}^{\infty} b_k} \leq \frac{\delta \left(1 + \frac{\gamma}{p+\beta}\right)^m \left[1 - \alpha(p+1) + (A + B)\right]a_{p+1}}{p+1}.$$

(19)
Thus by Definition 3, \( f \in R_p^n(\gamma, \beta, m, A, B, \alpha) \) for \( \eta \) given by (18). This completes the proof.

Now, we introduce the partial sums and the same property has been for other class in [8].

**Theorem 4**: Let \( f \in M_p \) be given by (2) and define the partial sums \( s_1(z) \) and \( s_q(z) \) by

\[
s_1(z) = z^p
\]

and

\[
s_q(z) = z^p + \sum_{k=p+1}^{p+q-1} a_k z^k, \quad q > p + 1.
\]

Suppose also that

\[
\sum_{k=p+1}^{\infty} d_k a_k \leq 1,
\]

\[
\left(d_k = \frac{\left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k-p) - \alpha(pk + (A + B))]b_k}{\alpha(p\lambda + (A + B))}\right).
\]

Thus, we have

\[
Re \left\{ \frac{f(z)}{s_q(z)} \right\} > 1 - \frac{1}{d_k}
\]

and

\[
Re \left\{ \frac{s_q(z)}{f(z)} \right\} > 1 - \frac{d_k}{1 + d_k}
\]

Each of the bounds in (22) and (23) is the best possible for \( p \in N \).

**Proof**: For the coefficients \( d_k \) given by (21), it is not difficult to verify that

\[
d_{k+1} > d_k > 1, \quad k = p + 1, p + 2, \cdots.
\]

Therefore, by using the hypothesis (20), we have

\[
\sum_{k=p+1}^{p+q-1} a_k + \sum_{k=p+1}^{\infty} d_k a_k \leq \sum_{k=p+1}^{\infty} d_k a_k \leq 1.
\]

By setting

\[
g_1 = d_k \left( \frac{f(z)}{s_q(z)} - \left(1 - \frac{1}{d_k}\right) \right) = 1 + \frac{d_k \sum_{k=p+q}^{\infty} a_k z^{k-p}}{1 + \sum_{k=p+q}^{p+q-1} a_k z^{k-p}}
\]
and applying (24) we find that

\[
\frac{|g_1(z) - 1|}{|g_1(z) + 1|} \leq \frac{d_k \sum_{k=p+q}^{\infty} a_k}{2 - 2 \sum_{k=p+1}^{p+q-1} a_k - d_k \sum_{k=p+1}^{\infty} a_k} \leq 1.
\]

This prove (22). Therefore, \( Re(g_1(z)) > 0 \) and we obtain

\[
Re \left\{ \frac{f(z)}{s_q(z)} \right\} > 1 - \frac{1}{d_k}.
\]

Now, in the same manner, we can prove the assertion (23), by setting

\[
g_2(z) = (1 + d_k) \left( \frac{s_q(z)}{f(z)} - \frac{d_k}{1 + d_k} \right).
\]

This completes the proof.

5 Radii of Starlikeness, Convexity and Close-to-Convexity

Using the inequalities (5), (6) and (7) and Theorem1, we can compute the radii of starlikeness, convexity and close-to-convexity.

**Theorem 5** : If \( f \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \), then \( f(z) \) is \( p \)-valently starlike of order \( \rho \) \((0 \leq \rho < p)\) in the disc \(|z| < r_1\), where

\[
r_1(\gamma, \beta, m, \lambda, A, B, \alpha, \rho) = \inf_k \left[ \left( \frac{p - \rho}{\alpha(k - \rho)} \right) \left( \frac{(k - p)^m}{\alpha(k - \rho)(p + \beta)} \right) \right]^{\frac{1}{p - \rho}}.
\]

**Proof** : It is sufficient to show that

\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \rho \quad (0 \leq \rho < p),
\]

for

\[
|z| < r_1(\gamma, \beta, m, \lambda, A, B, \alpha, \rho),
\]

we have

\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{(k - p)a_k|z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k|z|^{k-p}}.
\]
Thus

\[ \left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \rho, \]

if

\[ \sum_{k=p+1}^{\infty} \frac{(k - \rho)a_k|z|^{k-p}}{(p - \rho)} \leq 1. \]  \hspace{1cm} (26)

Hence, by Theorem 1, (26) will be true if

\[ \frac{(k - \rho)a_k|z|^{k-p}}{(p - \rho)} \leq \frac{1 + \frac{(k-p)\gamma}{(p+\beta)}}{\alpha[p\lambda + (A + B)]} m [(k - p) - \alpha(\lambda k + (A + B))] \]

and hence

\[ |z| \leq \left[ \frac{(p - \rho) \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]}{\alpha(k-p)[p\lambda + (A + B)]} \right]^{\frac{1}{k-p}}. \]

Setting \( |z| = r_1 \), we get the desired result.

**Theorem 6**: If \( f(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then \( f(z) \) is \( p \)-valently convex of order \( \rho \) \((0 \leq \rho < p)\) in the disc \( |z| < r_2 \), where

\[ r_2(\gamma, \beta, m, \lambda, A, B, \alpha, \rho) = \inf_k \left[ \frac{(p - \rho) \left(1 + \frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]}{\alpha k[p\lambda + (A + B)]} \right]^{\frac{1}{k-p}}. \]

**Proof**: It is sufficient to show that

\[ \left| 1 + \frac{f''(z)}{f'(z)} - p \right| \leq p - \rho, \quad (0 \leq \rho < p), \]

for

\[ |z| < r_2(\gamma, \beta, m, \lambda, A, B, \alpha, \rho), \]

we have

\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| = \frac{\sum_{k=p+1}^{\infty} k(k - p)a_k|z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} ka_k|z|^{k-p}}. \]

Thus

\[ \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \rho, \]

if

\[ \sum_{k=p+1}^{\infty} \frac{k(k - \rho)a_k|z|^{k-p}}{(p - \rho)} \leq 1. \]  \hspace{1cm} (27)
Hence, by Theorem 1, (27) will be true if
\[ \frac{k(k - \rho)a_k |z|^{k-p}}{(p - \rho)} \leq \frac{1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]}{\alpha[p\lambda + (A + B)]}, \]
and hence,
\[ |z| \leq \left(\frac{p - \rho}{(p - \rho)} \left(1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))] \right)^{\frac{1}{k-p}} \right) \frac{1}{\alpha k[k(p - \rho)][p\lambda + (A + B)]}. \]

Setting \( |z| = r_2 \), we get the desired result.

**Theorem 7**: Let a function \( f(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then \( f(z) \) is \( p \)-valently close-to-convex of order \( \rho \) \((0 \leq \rho < p)\) in the disc \( |z| < r_3 \), where
\[ r_3(\gamma, \beta, m, \lambda, A, B, \alpha, \rho) = \inf_k \left[ \frac{1}{\alpha k[k(p - \rho)][p\lambda + (A + B)]} \left(\frac{p - \rho}{(p - \rho)} \left(1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))] \right)^{\frac{1}{k-p}} - 1 \right) \right]. \]

**Proof**: It is sufficient to show that
\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \rho, \quad (0 \leq \rho < p), \]
for
\[ |z| < r_3(\gamma, \beta, m, \lambda, A, B, \alpha, \rho), \]
we have
\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} ka_k |z|^{k-p}. \]

Thus
\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \rho, \]
if
\[ \sum_{k=p+1}^{\infty} ka_k |z|^{k-p} \leq \frac{(p - \rho) \left(1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))] \right)^{\frac{1}{k-p}}}{\alpha k[k(p - \rho)][p\lambda + (A + B)]}. \] (28)

Hence, by Theorem 1, (28) will be true if
\[ \frac{k|z|^{k-p}}{(p - \rho)} \leq \frac{1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]}{\alpha[p\lambda + (A + B)]}, \]
and hence
\[ |z| \leq \left[ \frac{(p - \rho) \left(1 + \left(\frac{(k-p)\gamma}{(p+\beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))] \right)^{\frac{1}{k-p}}}{\alpha k[k(p - \rho)][p\lambda + (A + B)]} \right]. \]

Setting \( |z| = r_3 \), we get the desired result.
6 Weighted Mean and Arithmetic Mean

**Definition 4**: Let \( f_1 \) and \( f_2 \) be in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then the weighted mean \( w_q \) of \( f_1 \) and \( f_2 \) is given by
\[
w_q(z) = \frac{1}{2}[(1 - q)f_1(z) + (1 + q)f_2(z)], \quad 0 < q < 1.
\]

**Theorem 8**: Let \( f_1 \) and \( f_2 \) be in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then the weighted mean \( w_q \) of \( f_1 \) and \( f_2 \) is also in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \).

**Proof**: By Definition 4, we have
\[
w_q(z) = \frac{1}{2}[(1 - q)f_1(z) + (1 + q)f_2(z)]
\]
\[
= \frac{1}{2}\left[(1 - q) \left(z^p + \sum_{k=p+1}^{\infty} a_{k,1}z^k\right) + (1 + q) \left(z^p + \sum_{k=p+1}^{\infty} a_{k,2}z^k\right)\right]
\]
\[
= z^p + \sum_{k=p+1}^{\infty} \frac{1}{2}[(1 - q)a_{k,1} + (1 + q)a_{k,2}]z^k.
\]

Since \( f_1 \) and \( f_2 \) are in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \) so by Theorem 1, we get
\[
\sum_{k=p+1}^{\infty} \left(1 + \frac{(k - p)\gamma}{(p + \beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]a_{k,1}b_k \leq \alpha[p\lambda + (A + B)].
\]

And
\[
\sum_{k=p+1}^{\infty} \left(1 + \frac{(k - p)\gamma}{(p + \beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]a_{k,2}b_k \leq \alpha[p\lambda + (A + B)].
\]

Hence
\[
\sum_{k=p+1}^{\infty} \left(1 + \frac{(k - p)\gamma}{(p + \beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))] \times
\]
\[
\left(\frac{1}{2}[(1 - q)a_{k,1} + (1 + q)a_{k,2}]\right) b_k z^k
\]
\[
= \frac{1}{2}(1 - q) \sum_{k=p+1}^{\infty} \left(1 + \frac{(k - p)\gamma}{(p + \beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]a_{k,1}b_k
\]
\[
+ \frac{1}{2}(1 + q) \sum_{k=p+1}^{\infty} \left(1 + \frac{(k - p)\gamma}{(p + \beta)}\right)^m [(k - p) - \alpha(\lambda k + (A + B))]a_{k,2}b_k
\]
\[
\leq \frac{1}{2}(1 - q)\alpha[p\lambda + (A + B)] + \frac{1}{2}(1 + q)\alpha[p\lambda + (A + B)] = \alpha[p\lambda + (A + B)].
\]
Therefore \( w_q \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \).
The proof is complete.

**Theorem 9**: Let \( f_1(z), f_2(z), \ldots, f_l(z) \) defined by

\[
 f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i}z^k, \quad (a_{k,i} \geq 0, i = 1, 2, \ldots, l, \ k \geq p + 1) \tag{30}
\]

be in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then the arithmetic mean of \( f_i(z) \) \((i = 1, 2, \ldots, l)\) defined by

\[
 h(z) = \frac{1}{l} \sum_{i=1}^{l} f_i(z) \tag{31}
\]

is also in the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \).

**Proof**: By (30), (31), we can write

\[
 h(z) = \frac{1}{l} \sum_{i=1}^{l} \left( z^p + \sum_{k=p+1}^{\infty} a_{k,i}z^k \right)
\]

\[
 = z^p + \sum_{k=p+1}^{\infty} \left( \frac{1}{l} \sum_{i=1}^{l} a_{k,i} \right) z^k.
\]

Since \( f_i \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \) for every \((i = 1, 2, \ldots, l)\) so by using Theorem 1, we prove that

\[
 \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m ((k-p) - \alpha(\lambda k + (A + B)) (\frac{1}{l} \sum_{i=1}^{l} a_{k,i}) b_k
\]

\[
 = \frac{1}{l} \sum_{i=1}^{l} \left( \sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m ((k-p) - \alpha(\lambda k + (A + B)))a_{k,i}b_k
\]

\[
 \leq \frac{1}{l} \sum_{i=1}^{l} \alpha[p\lambda + (A + B)]
\]

\[
 = \alpha[p\lambda + (A + B)].
\]

7 Linear Combination

In the theorem below, we prove a linear combination for the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \).

**Theorem 10**: Let

\[
 f_i(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,i}z^k, \quad (a_{k,i} \geq 0, i = 1, 2, \ldots, l, k \geq p + 1)
\]
belong to the class \( R_p(\gamma, \beta, m, \lambda, A, B, \alpha) \). Then

\[
F(z) = \sum_{i=1}^{l} c_i f_i(z) \in R_p(\gamma, \beta, m, \lambda, A, B, \alpha),
\]

where

\[
\sum_{i=1}^{l} c_i = 1.
\]

**Proof**: By Theorem 1, we can write for every \( i \in \{1, 2, \ldots, l\} \)

\[
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m \frac{[(k-p) - \alpha(\lambda k + (A + B))]}{\alpha[p\lambda + (A + B)]} a_{k,i} b_k \leq 1.
\]

Therefore

\[
F(z) = \sum_{i=1}^{l} c_i \left( z^p + \sum_{k=p+1}^{\infty} a_{k,i} z^k \right) = z^p + \sum_{k=p+1}^{\infty} \left( \sum_{i=1}^{l} c_i a_{k,i} \right) z^k.
\]

Hence

\[
\sum_{k=p+1}^{\infty} \left( 1 + \frac{(k-p)\gamma}{(p+\beta)} \right)^m \frac{[(k-p) - \alpha(\lambda k + (A + B))]}{\alpha[p\lambda + (A + B)]} \left( \sum_{i=1}^{l} c_i a_{k,i} \right) b_k \leq 1.
\]

Then \( F(z) \in R_p(\gamma, \beta, m, A, B, \alpha) \). So the proof is complete.

**References**


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