A Study of a Two Variables Gegenbauer Matrix Polynomials and Second Order Matrix Partial Differential Equations. A comment

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Abstract

In this comment we will demonstrate that one of the main formulas given in Ref. [9] is incorrect.

Mathematics Subject Classification: 33C45, 42C05

Keywords: Generating function, matrix polynomials, two variable Gegenbauer polynomials, Gegenbauer matrix polynomials

1 Introduction and motivation

A family of orthogonal polynomials \( \{P_n(x)\}_{n \geq 0} \) can be associated with so-called “generating functions”, which are a useful and invaluable tool for studying this class of functions. Usually, a generating function is a function of two variables \( F(x,t) \) and analytic on some set \( D \in \mathbb{C}^2 \), so that

\[
F(x,t) = \sum_{n=0}^{\infty} \alpha_n P_n(x)t^n, (x,t) \in D.
\]

For example, see [10], if \( r_1 \) and \( r_2 \) are the roots of the quadratic equation \( 1 - 2xt + t^2 = 0 \), and if \( r \) is the minimum of the set \( \{ |r_1|, |r_2| \} \), then, for a parameter \( \lambda \) the function \( F(x,t) = (1 - 2xt + t^2)^{-\lambda} \), regarded as a function of \( t \), is analytic in the disk \( |t| < r \).
From complex variable theory, we have the following generating function for the Gegenbauer polynomials \( C_n^\lambda(x) \):

\[
F(x, t) = \left( 1 - 2xt + t^2 \right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x)t^n, \quad |t| < r, |x| < 1.
\]

The analogous extension to the matrix framework for the classical case of Hermite [8], Jacobi [3], Gegenbauer [7], Laguerre [6] and Chebyshev [2] polynomials has been carried out in recent years, and properties and applications of different classes for these matrix polynomials are the focus of a number of previous papers, see [1, 4, 12, 11, 5] and references therein, for example.

In the matrix case, the importance of the generating function is similar to the scalar case, taking into account the possible spectral restrictions (for a matrix \( A \in \mathbb{C}^{N \times N} \) we will denote by \( \sigma(A) \) the matrix spectrum \( \sigma(A) = \{ z; z \text{ is an eigenvalue of } A \} \)).

For example:

- **LAGUERRE MATRIX POLYNOMIALS.** If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) such that \(-k \not\in \sigma(A)\) for every integer \( k > 0 \), and \( \lambda \) is a complex number with \( \text{Re}(\lambda) > 0 \), the generating function [6] is given by:

\[
(1 - t)^{-(A+I)} \exp\left( \frac{-\lambda xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n, \quad \forall \ x, t \in \mathbb{C}, |t| < 1.
\]

- **HERMITE MATRIX POLYNOMIALS.** If \( A \) is a matrix in \( \mathbb{C}^{N \times N} \) such that \( \text{Re}(z) > 0, \forall z \in \sigma(A) \) (i.e. \( A \) is positive stable), the generating function [8] is given by

\[
e^{xt\sqrt{A-t^2}I} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, A) t^n, \quad (x, t) \in \mathbb{R}^2.
\]

2 **The detected mistake. An example**

Recently, in Ref. [9], a new extension of Gegenbauer matrix polynomials with two variables was presented. The starting point for their definition was the following double generating formula (8):

\[
(1 - 2xs + s^2 - 2yt + t^2)^{-A} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} C_{n,k}^A(x, y) s^n t^k \quad (8)
\]

This formula turns out to be the key for the definition and development of the properties mentioned in the paper [9], with the intention to guarantee that (8) is term-wise differentiable with respect to its variables \( x, s, y, t \). However, we will see
that formula (8) is incorrect. For this, first we note that for a matrix $A$ exponent one has to define

$$t^A = e^{A \log t}.$$ 

Obviously, $t^A$ only makes sense if $t \neq 0$. Thus, expression (8) is void of meaning if the term $(1 - 2xs + s^2 - 2yt + t^2)$ vanishes. Assuming, for example, the values

$$x = y = 1, s = t = \frac{1}{2} \left(2 \pm \sqrt{2}\right),$$

the term $(1 - 2xs + s^2 - 2yt + t^2)$ is zero and (8) is meaningless.

Therefore, I ask the author of Ref. [9] to clarify the domain of choice for the variables $x, y, t, s$ in formula (8) in order to guarantee the validity of the remaining formulas which are derived from (8) and used in the remainder of the aforementioned paper.

### 3 Remark

I can not end this commentary without adding a brief remark. The introduction of Gegenbauer matrix polynomials was made in Ref. [7]. In Ref. [9] the basic reference given is [13]. However, reference [13] appeared nine years later, and several of the formulas in [13] are incorrect. As a single example, we will consider formula (5) of [9] (which is the same formula (13) in [13]):

$$F = (1 - 2xt + t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n \quad (5)$$

where $A$ is a positive definite matrix in $\mathbb{C}^{N \times N}$ ($\text{Re}(z) > 0, \forall z \in \sigma(A)$).

As already mentioned, expression (5) is meaningless if the term $1 - 2xt + t^2$ vanishes. Assuming, for example, the different values

- $x = t = 1, (i.e. |x| = 1, |t| = 1),$
- $x = \frac{i}{\sqrt{10}}, t = -i\left(\sqrt{11} - 1\right), (i.e. |x| = 0.316 < 1, |t| = 0.733 < 1, t^2 = -1),$
- $x = 2, t = 2 + \sqrt{3}, (i.e. |x| > 1, |t| = 3.73205 > 1),$

the term $1 - 2xt + t^2$ is zero in the three cases and (5) is meaningless.

### References


Received: February, 2012