Convergence Theorems for a Pair of Asymptotically and Multi-Valued Nonexpansive Mapping in Banach Spaces

Kritsana Sokhuma

Department of Mathematics, Faculty of Science and Technology Muban Chom Bueng Rajabhat University, Ratchaburi 70150, Thailand

k.sokhuma@yahoo.co.th

Abstract

We establish the convergence theorems of the modified Ishikawa iteration methods with respect to a pair of the single valued asymptotically nonexpansive mapping $t$ and the multi-valued nonexpansive mapping $T$. This results we obtain are analogs of Banach space results of Sokhuma [7], Sokhuma and Keawkhao [6].

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1 Introduction

Let $X$ be a Banach space and $E$ a nonempty subset of $X$. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of $E$, by $F(E)$ the family of the closed subset of $E$, by $K(E)$ the family of nonempty compact subsets of $E$, by $FC(E)$ the family of nonempty closed convex subsets of $E$, and by $KC(E)$ the family of nonempty compact convex subsets of $E$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}, \ A, B \in FB(X),$$
where $\text{dist}(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point $a$ to the subset $B$.

A multi-valued mapping $T : E \to F(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \text{ for all } x, y \in E.$$  

(1)

A point $x$ is a fixed point for a multi-valued mapping $T$ if $x \in Tx$.

From now on, recall that $\text{Fix}(T)$ stands for the set of fixed points of a mapping $T$ and $\text{Fix}(t) \cap \text{Fix}(T)$ stands for the set of common fixed points of $t$ and $T$. Precisely, a point $x$ is called a common fixed point of $t$ and $T$ if $x = tx \in Tx$.

A Banach space $X$ is said to satisfy the Opial condition if whenever a sequence $\{x_n\}$ in $X$ converges weakly to $x_0 \in X$, then

$$\liminf_{n \to \infty} \|x_n - x_0\| < \liminf_{n \to \infty} \|x_n - x\| \text{ for all } x \in X, \; x \neq x_0.$$  

(2)

The Opial condition plays an important role in convergence of sequences and in the study of the demiclosedness principle of nonlinear mappings of Banach spaces.

One of the most celebrated results about multi-valued mappings was given, by using Edelstein’s method of asymptotic centers (see [1]), by T. C. Lim in 1974.

**Theorem 1.1.** ([4]) Let $E$ be a nonempty closed bounded convex subset of a uniformly convex Banach space $X$ and $T : E \to K(E)$ a nonexpansive mapping. Then $T$ has a fixed point.

Asymptotic fixed point theorems are those theorems involving the existence of fixed points of a mapping $t : X \to X$ derived from the behavior of the iterates $t^n$ for large $n$. A mapping $t : E \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n \to \infty} k_n = 1$ such that

$$\|t^n x - t^n y\| \leq k_n \|x - y\|$$

for all $x, y \in E$ and $n \in \mathbb{N}$.

In 2010, Sokhuma [7] proved the following theorem.

**Theorem 1.2.** ([7])Let $X$ be a uniformly convex Banach space, $E$ a nonempty bounded closed convex subset of $X$, and $t : E \to E$ and $T : E \to KC(E)$ is an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition $t$ and $T$ are commuting. Then $t$ and $T$ have a common fixed point, i.e., there exists a point $x$ in $E$ such that $x = tx \in Tx$. 
The plan of the paper is as follows. First, we introduce an iteration process in a new sense, called the modified Ishikawa iteration method with respect to a pair of asymptotically nonexpansive mapping and multi-valued nonexpansive mappings in a nonempty compact convex subset of a uniformly convex Banach space. Finally, we establish the convergence theorems of the modified Ishikawa iteration methods.

2 Basic Properties

In this section, we discuss about the characterization of a uniformly convex Banach space and some basic properties on such space. Thereafter, a new iteration method is introduced. The characterization of a uniformly convex Banach space we use is the following:

**Proposition 2.1.** ([9]) A Banach space $X$ is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(s) = 0 \iff s = 0$, such that

$$
\| \lambda x + (1 - \lambda)y \|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\varphi(\|x - y\|)
$$

for all $\lambda \in [0,1]$ and all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

The following lemma was proved by Schu (cf.[5]) in 1991.

**Lemma 2.2.** ([5]) Let $X$ be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of $X$ such that $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$ and $\lim_{n \to \infty} \|u_n x_n + (1 - u_n)y_n\| = a$ for some $a \geq 0$. Then, $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

The following fact is well-known.

**Lemma 2.3.** Let $X$ be a Banach space and $E$ be a nonempty compact convex subset of $X$ and $\{x_n\}$ be the sequence in $E$. Then,

$$
\operatorname{dist}(y, Ty) \leq \|y - x_n\| + \operatorname{dist}(x_n, Tx_n) + H(Tx_n, Ty)
$$

where $y \in E$ and $T$ be a multi-valued mapping from $E$ into $\mathbb{F}B(E)$.

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle.

**Definition 2.4.** ([3]) Let $X$ be a Banach space and $K \subseteq X$. A mapping $f : K \to X$ is demiclosed (at $y$) if the conditions $\{x_j\}$ converges weakly to $x$ and $\{f(x_j)\}$ converges strongly to $y$ then $x \in K$ and $f(x) = y$. 
Theorem 2.5. ([3]) Let $X$ be a uniformly convex Banach space and $E$ be a nonempty bounded closed convex subset of $X$, and $t : E \to E$ be an asymptotically nonexpansive mapping. Then $I - t$ is demiclosed at 0.

The important property of a uniformly convex Banach space we use in the following lemma can be founded in [8].

Lemma 2.6. Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be a sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$ 

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n \to \infty} a_n = 0$.

In 1974, Ishikawa introduced the following well-known iteration.

Definition 2.7. (see [2]). Let $X$ be a Banach space, let $E$ be a closed convex subset of $X$, and let $t$ be a self map on $E$. For $x_0 \in E$, the sequence $\{x_n\}$ of Ishikawa iterates of $t$ is defined by

$$y_n = (1 - \beta_n)x_n + \beta_n tx_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n ty_n, \quad n \geq 0,$$ \hspace{1cm} (3)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

Now, let us introduce a new iteration method modifying the above ones and call it the modified Ishikawa iteration method.

Definition 2.8. Let $E$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, $t : E \to E$, $T : E \to KC(E)$ an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that $t$ and $T$ are commuting. The sequence of the modified Ishikawa iterated is defined by $x_1 \in E$

$$y_n = (1 - \beta_n)x_n + \beta_n z_n,$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n t^n y_n, \quad n \geq 0,$$ \hspace{1cm} (4)

where $z_n \in Tt^n x_n$ and $\alpha_n, \beta_n \in [0, 1], \quad n \geq 1$.

3 Main Results

One of our main goals is to present convergence theorems for asymptotically nonexpansive mapping and multi-valued nonexpansive mapping. We first prove the following lemmas, which play very important roles in this section.
Lemma 3.1. Let $E$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, $t : E \to E$, $T : E \to KC(E)$ an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that $t$ and $T$ are commuting and $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$ satisfying $T w = \{w\}$ for all $w \in \text{Fix}(t) \cap \text{Fix}(T)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by (4). Then $\lim_{n \to \infty} \|x_n - w\|$ exists for all $w \in \text{Fix}(t) \cap \text{Fix}(T)$.

Proof. Let $x_1 \in E$ and $w \in \text{Fix}(t) \cap \text{Fix}(T)$, we have

$$\|x_{n+1} - w\| = \|(1 - \alpha_n) (x_n - w) + \alpha_n (t^n y_n - t^n w)\|$$

$$\leq (1 - \alpha_n) \|x_n - w\| + \alpha_n \|t^n y_n - t^n w\|$$

$$\leq (1 - \alpha_n) \|x_n - w\| + \alpha_n k_n \|y_n - w\|$$

$$= (1 - \alpha_n) \|x_n - w\| + \alpha_n k_n (1 - \beta_n) \|x_n - w\| + \alpha_n k_n \beta_n \|z_n - w\|$$

$$\leq (1 - \alpha_n) \|x_n - w\| + \alpha_n k_n (1 - \beta_n) \|x_n - w\| + \alpha_n k_n \beta_n \|z_n - w\|$$

$$= (1 - \alpha_n) \|x_n - w\| + \alpha_n k_n (1 - \beta_n) \|x_n - w\| + \alpha_n k_n \beta_n \|z_n - w\|$$

By the convergence of $k_n$ and $\alpha_n$, $\beta_n \in (0, 1)$, then there exists some $M > 0$ such that

$$\|x_{n+1} - w\| \leq [1 + M(k_n - 1)] \|x_n - w\|.$$

By condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and Lemma 2.6, we know that $\lim_{n \to \infty} \|x_n - w\|$ exists.

Lemma 3.2. Let $E$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, $t : E \to E$, $T : E \to KC(E)$ an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that $t$ and $T$ are commuting and $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$ satisfying $T w = \{w\}$ for all $w \in \text{Fix}(t) \cap \text{Fix}(T)$ and $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by (4). Then $\lim_{n \to \infty} \|t^n y_n - x_n\| = 0.$
Proof. From Lemma 3.1, we setting \( \lim_{n \to \infty} \|x_n - w\| = c \).
Consider,
\[
\|y_n - w\| = \|(1 - \beta_n)(x_n - w) + \beta_n(z_n - w)\|
\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|z_n - w\|
= (1 - \beta_n)\|x_n - w\| + \beta_n\text{dist}(w, Tt^n x_n)
= (1 - \beta_n)\|x_n - w\| + \beta_n H(Tw, Tt^n x_n)
\leq (1 - \beta_n)\|x_n - w\| + \beta_n\|t^n x_n - w\|
\leq (1 - \beta_n)\|x_n - w\| + \beta_n k\|x_n - w\|.
\]
We have
\[
\|t^n y_n - w\| \leq k_n\|y_n - w\|
\leq k_n[(1 - \beta_n)\|x_n - w\| + \beta_n k_n\|x_n - w\|]
= k_n (1 - \beta_n)\|x_n - w\| + \beta_n k^2_n\|x_n - w\|
= (k_n - k_n \beta_n + \beta_n k_n^2)\|x_n - w\|
= [k_n + \beta_n k_n(k_n - 1)]\|x_n - w\|
\leq [1 + \beta_n k_n(k_n - 1)]\|x_n - w\|.
\]
Then we have,
\[
\limsup_{n \to \infty} \|t^n y_n - w\| \leq \limsup_{n \to \infty} k_n\|y_n - w\| \leq \limsup_{n \to \infty} [1 + \beta_n k_n(k_n - 1)]\|x_n - w\|.
\]
By \( k_n \to 1 \) as \( n \to \infty \) and \( \alpha_n, \beta_n \in (0, 1) \), which implies that
\[
\lim_{n \to \infty} \|t^n y_n - w\| \leq \lim_{n \to \infty} \|y_n - w\| \leq \lim_{n \to \infty} \|x_n - w\| = c.
\quad (5)
\]
Since, \( c = \lim_{n \to \infty} \|x_{n+1} - w\| = \lim_{n \to \infty} \| (1 - \alpha_n)(x_n - w) + \alpha_n(t^n y_n - w) \|.
\]
Then by condition of \( \alpha_n \) and Lemma 2.2, we have
\[
\lim_{n \to \infty} \|t^n y_n - x_n\| = 0.
\]

Lemma 3.3. Let \( E \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( X \), \( t : E \to E \), \( T : E \to KC(E) \) an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that \( t \) and \( T \) are commuting and \( \text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset \) satisfying \( Tw = \{w\} \) for all \( w \in \text{Fix}(t) \cap \text{Fix}(T) \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let \( \{x_n\} \) be the sequence of the modified Ishikawa iterates defined by (4). Then \( \lim_{n \to \infty} \|x_n - z_n\| = 0. \)
Proof. Consider,
\[ \|x_{n+1} - w\| = \|(1 - \alpha_n)(x_n - w) + \alpha_n(t^ny_n - w)\| \]
\[ \leq (1 - \alpha_n)\|x_n - w\| + \alpha_n\|t^ny_n - w\| \]
\[ \leq (1 - \alpha_n)\|x_n - w\| + \alpha_n k_n\|y_n - w\| \]
and hence
\[ \frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq k_n\|y_n - w\| - \|x_n - w\|. \]
Therefore, since \(0 < a \leq \alpha_n \leq b < 1\),
\[ \left( \frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \right) + \|x_n - w\| \leq k_n\|y_n - w\|. \]
Thus,
\[ \liminf_{n \to \infty} \left\{ \left( \frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \right) + \|x_n - w\| \right\} \leq \liminf_{n \to \infty} k_n\|y_n - w\|. \]
It follow that
\[ c \leq \liminf_{n \to \infty} \|y_n - w\|. \]
Since, from (5), \(\limsup_{n \to \infty} \|y_n - w\| \leq c\), we have
\[ c = \lim_{n \to \infty} \|y_n - w\| = \lim_{n \to \infty} \|(1 - \beta_n)(x_n - w) + \beta_n(z_n - w)\|. \]
Recall that,
\[ \|z_n - w\| = \text{dist} (z_n, Tw) \leq H(Tt^nx_n, Tw) \leq \|t^nx_n - w\| \leq k_n\|x_n - w\|. \]
Hence we have
\[ \limsup_{n \to \infty} \|z_n - w\| \leq \limsup_{n \to \infty} k_n\|x_n - w\| = \limsup_{n \to \infty} \|x_n - w\| = c. \quad (6) \]
Thus,
\[ \lim_{n \to \infty} \|x_n - z_n\| = 0. \]

Lemma 3.4. Let \(E\) be a nonempty bounded closed convex subset of a uniformly convex Banach space \(X\), \(t : E \to E\), \(T : E \to \text{KC}(E)\) an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that \(t\) and \(T\) are commuting and \(\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset\) satisfying \(Tw = \{w\}\) for all \(w \in \text{Fix}(t) \cap \text{Fix}(T)\) and \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\). Let \(\{x_n\}\) be the sequence of the modified Ishikawa iterates defined by (4). Then \(\lim_{n \to \infty} \|t^nx_n - x_n\| = 0\).
Proof. It is easy to see that, \( \|t^nx_n - x_n\| \leq k_n \beta_n \|z_n - x_n\| + \|t^ny_n - x_n\| \).
Then we have
\[
\lim_{n \to \infty} \|t^nx_n - x_n\| \leq \lim_{n \to \infty} k_n \beta_n \|z_n - x_n\| + \lim_{n \to \infty} \|t^ny_n - x_n\|.
\]
Hence, by Lemma 3.2 and Lemma 3.3, \( \lim_{n \to \infty} \|t^nx_n - x_n\| = 0 \). \( \square \)

Lemma 3.5. Let \( E \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( X \), \( t : E \to E \), \( T : E \to KC(E) \) an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that \( t \) and \( T \) are commuting and \( \text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset \) satisfying \( Tw = \{w\} \) for all \( w \in \text{Fix}(t) \cap \text{Fix}(T) \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let \( \{x_n\} \) be the sequence of the modified Ishikawa iterates defined by (4). Then \( \lim_{n \to \infty} \|tx_n - x_n\| = 0 \).

Proof. Consider,
\[
\|tx_n - x_n\| \leq \|x_n - t^n x_n\| + \|t^n x_n - t x_n\|
\]
\[
\leq \|x_n - t^n x_n\| + k_1 \left( \|t^{n-1} x_n - t^{n-1} x_{n-1} + t^{n-1} x_{n-1} - x_n\| \right)
\]
\[
\leq \|x_n - t^n x_n\| + k_1 k_{n-1} \|x_n - x_{n-1}\| + k_1 \|t^{n-1} x_{n-1} - x_{n-1}\|
\]
\[
\leq \|x_n - t^n x_n\| + k_1 k_{n-1} \|x_n - x_{n-1}\| + k_1 k_{n-1} (1 - \alpha_{n-1}) \|x_{n-1} - t^{n-1} x_{n-1}\|
\]
\[
+ k_1 k_{n-1} \alpha_{n-1} \|y_{n-1} - x_{n-1}\|.
\]
It follows from Lemma 3.2 - 3.4, we have \( \lim_{n \to \infty} \|tx_n - x_n\| = 0 \). \( \square \)

Theorem 3.6. Let \( E \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( X \) which satisfies the Opial’s condition, \( t : E \to E \), \( T : E \to KC(E) \) an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that \( t \) and \( T \) are commuting and \( \text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset \) satisfying \( Tw = \{w\} \) for all \( w \in \text{Fix}(t) \cap \text{Fix}(T) \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let \( \{x_n\} \) be the sequence of the modified Ishikawa iterates defined by (4). Then sequence \( \{x_n\} \) converges weakly to a common fixed point.

Proof. It follow from the boundedness of \( \{x_n\} \), since \( X \) is uniformly convex, every bounded subset of \( X \) is weakly compact, so that there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) converge weakly to a point \( p \in E \). Therefore, it follows
from Lemma 3.5 that

\[ \lim_{n \to \infty} \| t x_{n_i} - x_{n_i} \| = 0. \]

By Theorem 2.5, we know \( I - t \) is demiclosed, so that \( p \in \text{Fix}(t) \).

Consider,

\[ d(p, Tp) \leq \| x_{n_i} - p \| + \text{dist}(x_{n_i}, T t^{n_i} x_{n_i}) + H(T t^{n_i} x_{n_i}, Tp) \]
\[ \leq \| x_{n_i} - p \| + \| x_{n_i} - z_{n_i} \| + \| t^{n_i} x_{n_i} - p \| \]
\[ \leq \| x_{n_i} - p \| + \| x_{n_i} - z_{n_i} \| + k_{n_i} \| x_{n_i} - p \| \to 0, \text{ as } n_i \to \infty. \]

Therefore, \( p \in Tp \) and hence \( p \in \text{Fix}(t) \cap \text{Fix}(T) \).

Finally, we prove that the sequence \( \{ x_n \} \) converge weakly to \( p \). In fact, suppose this is not true, then there must exists a subsequence \( \{ x_{n_i} \} \subset \{ x_n \} \) such that \( \{ x_{n_i} \} \) converges weakly to \( q \in E, p \neq q \). Then, by the same method given above, we can prove that \( q \in \text{Fix}(t) \), the limit \( \lim_{n \to \infty} \| x_n - p \| \) exists. Then we can let

\[ \lim_{n \to \infty} \| x_n - p \| = d_1, \quad \lim_{n \to \infty} \| x_n - q \| = d_2. \]

By Opial's condition of \( X \), we have

\[ d_1 = \limsup_{k \to \infty} \| x_{n_k} - p \| < \limsup_{k \to \infty} \| x_{n_k} - q \| = d_2 \]
\[ = \limsup_{k \to \infty} \| x_{n_k} - q \| < \limsup_{k \to \infty} \| x_{n_k} - p \| = d_1. \]

This is contradiction, hence \( p = q \). This implies that \( \{ x_n \} \) converges weakly to \( p \).

Therefore \( \{ x_n \} \) converges weakly to a common fixed point.

\( \Box \)

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