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A Note of the Mcshane Integral on the Riesz Space

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Abstract

The main goal of this article is to investigate the integration theory of Mcshane type for functions with values in ordered spaces. Afected by the work of Boccuto, Riecan, and Vrábelová with Kurzweil-Henstock integration we studied the same problems for another important type of integration on such space as Mcshane ones.In this paper we present in other way the definition of Mcshane integral on the Riesz space using a very important lemma of famous Fremlin. In the second section we reconstruct allmost all the propeties of Mchane integral given in [5], [6] and these ones become a little more stronger. We arrive some new results compared with Henstock-Kurzweil ones. In the third section we define the strong version of Mcshane integral and give the neccesary and sufficent condition of this concept. In the fourth section we prove the fundamental theorems of Calculus for the *DM*-integral and in the fifth we extended an application of this integration to Walsh series .

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1. Introduction

In this paper we adopted the terminology used from [1], [2],[4] , [5] [7] and [8]. Let $\mathbb{N}$ be the set of all strictly positive integers,$\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ be the set of all strictly positive real numbers, $\widetilde{\mathbb{R}}$ the set of all extended reals numbers .We begin with some preliminary definitions and results.

Definition1.1 A Riesz space R is said to be Dedekind complete if every nonempty subset of R,bounded from above, has supremum in R.

Definition 1.2 A bounded double sequence $\left(a_{i,j}\right)_{i,j}$ in R is called a (D)- sequence or regulator if $a_{i,j} \geq a_{i,j+1}$ $\forall$ i, j$\in \mathbb{N}$ and $\bigwedge_{j\in\mathbb{N}} a_{i,j} = 0$ $\forall$ i, $\in \mathbb{N}$. We say that $(r_n)_{n\in\mathbb{N}} \in$ R, (D)- converges to an r $\in$ R if there exists a (D)- sequence$\left(a_{i,j}\right)_{i,j}$ in R, satisfying the following condition

$\forall$ $\varphi: \mathbb{N} \to \mathbb{N}$, there exists an integer n_0 such that

$$|r_n - r| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

For all n$\geq$ n_0 . In this case, we write $(D)\lim_{n\to+\infty} r_n = r$.

Analogously, given $\ell \in$ R, a functionf: $A \to R$, where $\emptyset \neq A \subset \widetilde{\mathbb{R}}$,and a limit point x_0 for A, we will say that$(D)\lim_{x\to x_0} f(x) = \ell$ if there exists a (D)-sequence $\left(a_{i,j}\right)_{i,j}$ in R such that , $\forall$ $\varphi \in \mathbb{N}^{\mathbb{N}}$,there exists a neighborhood $\mathcal{U}$ of x_0 such that for all x $\in \mathcal{U} \cap A\backslash\{x_0\}$ we get

$$|f(x) - \ell| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)} \ .$$

Definition 1. 3 An Abelian partially ordered group (R,+, $\leq$) is called a lattice ordered group (or an l-group) if it is a lattice and following implication holds

$$[a \leq b] \Rightarrow [a + c \leq b + c] \quad \forall \text{ a,b,c} \in R$$

Definition 1. 4 An l-group R is said to be weakly σ $-$distributive if for every bounded double sequence $\left(b_{i,j}\right)_{i,j}$ with $b_{i,j} \geq b_{i,j+1}$ $\forall$ i,j one has

$\bigvee_{i\in 1}^{\infty}\left(\bigwedge_{j=i}^{\infty} b_{i,j}\right) = \bigwedge_{\varphi\in\phi}\left(\bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)}\right)$

Lemma 1.5 Let R be a Dedekind complete l-group ,and $\left(a_{i,j}^{(n)}\right)_{i,j}$, $n \in \mathbb{N}$ be a sequence of regulators in R.Then to every a$\in$ R , $a \geq 0$,there exists a regulator $\left(a_{i,j}\right)_{i,j}$ such that : $a \wedge \left(\sum_{n=1}^{k}\left(\bigvee_{i\in 1}^{\infty} a_{i,\varphi(i+n)}^{(n)}\right)\right) \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$ for every k$\in \mathbb{N}$ and $\varphi \in \phi$.

Definition 1.6 A subset A of a vector lattice $E \subset R$ is called solid if $x \in A$ and $y \in R$, inequation $|y| \leq |x|$ implies $y \in A$.

2. Definition of the Mcshane integral in Riesz space

Let $X = [a, b] \subset \mathbb{R}$. We shall work with a family $\mathcal{F}$ of all finite unions of closed subintervals such that $X \in \mathcal{F}$ and closed under the intersection and finite union. Denote λ a monotone and additive mapping $\lambda: \mathcal{F} \to [0, \infty]$.

The additivity means that:

$$\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B)$$

whenever $A, B, A \cup B \in \mathcal{F}$.

By a partition(detaily, $(\mathcal{F}, \lambda) -$ partition)of a set $A \in \mathcal{F}$ we mean a finite collection $\{(\mathcal{U}_1, t_1), \ldots (\mathcal{U}_k, t_k)\}$ such that

(i)$\mathcal{U}_1, \ldots \ldots \mathcal{U}_k \in \mathcal{F}$,

(ii) $\bigcup_{i=1}^{k} (\mathcal{U}_1) = A$,

(iii) $\lambda(\mathcal{U}_i \cap \mathcal{U}_j) = 0$ whenever $i \neq j$,

A finite collection$\{(\mathcal{U}_1, t_1), \ldots (\mathcal{U}_k, t_k)\}$ of subsets of $A \in \mathcal{F}$, satisfying conditions (i) , (iii) is said to be decomposition of A . We shall assume that $\mathcal{F}$ separates points in the following way: to any $A \in \mathcal{F}$ there exists a sequence $(\mathcal{A}_n)_n$ of partitions of A such that

(i)$\mathcal{A}_{n+1}$ is a refinement of $\mathcal{A}_n$,

(ii) to any $x, y \in A$, $x \neq y$, there exists $n \in \mathbb{N}$ and $B \in \mathcal{A}_n$ such that $x \in B$ and $y \notin B$.

A gauge on a set $A \subset X = [a, b]$ is a mapping δ assigning to every point $x \in A$ a neighborhood $\delta(x)$ of x . If $\Pi = \{(\mathcal{U}_1, t_1), \ldots (\mathcal{U}_k, t_k)\}$ is a decomposition of A and δ is a gauge on A , then we say that Π is $\delta -$ fine if $\mathcal{U}_i \subset \delta(t_i) =]t_i - \delta, t_i + \delta[$ for any $i \in \{1,2, \ldots, k\}$. If $X = [a, b] \subset \mathbb{R}$, with the usual topology , $\mathcal{F} =$ {the family of all finite unions of closed subintervals of X} , $\lambda([\alpha, \beta]) = \beta - \alpha$, $a \leq \alpha < \beta \leq b$ be Lebesgue measure , any gauge on $[\alpha, \beta]$ can be represented by a real function $d: [a, b] \to \mathbb{R}^+$, where we put $\delta(x) = \big(x - d(x), x + d(x)\big)$.

Let us return to the definition of Mcshane Integral on $X = [a, b]$. If $\Pi = \{(\mathcal{U}_1, t_1), \ldots (\mathcal{U}_k, t_k)\}$ is a decomposition of a set A ,and $f: [a, b] \to R$, then we define the Riemann sum as follows:

$$S(f,\Pi) = \sum_{i=1}^{k} f(t_i)\ |\mathcal{U}_i|$$

Where $|\mathcal{U}_i|$ is Lebesgue measure.

We note that the fact that $\mathcal{F}$ separates points guarantees the existence of at least one $\delta -$ fine partition Π such that $S(f,\Pi)$ is well-defined for any gauge δ.

Definition 2.1 A function $f : [a,b] \to R$ is said to be DM -integrable (in sens of Mcshane integral) on a set $A \subset [a,b]$ if there exist $I \in R$ and a (D)-sequence$\left(b_{i,j}\right)_{i,j}$such that to any , $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge δ on A such that

$$|S(f,\Pi) - I| \leq \vee_{i\in 1}^{\infty} b_{i,\varphi(i)}$$

whenever Π is a $\delta -$ fine partition of A such that $S(f,\Pi)$ exists in R . We denote

$$I = \int_A f$$

It is clear that (DM)-integral is Henstock –Kurzweil integral (see[1])

Proposition 2.2 Let $\left(a_{i,j}\right)_{i,j}, \left(b_{i,j}\right)_{i,j}$be two $(D) -$ sequences .Then there exists a $(D) -$sequences $\left(c_{i,j}\right)_{i,j}$, such that : $\vee_{i\in 1}^{\infty} a_{i,\varphi(i)} + \vee_{i\in 1}^{\infty} b_{i,\varphi(i)} \leq \vee_{i\in 1}^{\infty} c_{i,\varphi(i)}$ for any $\varphi \in \Phi$.

Lemma 2.3 The element I from Definition 2.1 is determined uniquely.

Similarly as in [1],[2] to prove the following Propositions.

Proposition 2.4 If f,g are integrable on$[a,b]$,and $\alpha \epsilon \mathbb{R}$, then $f + g, \alpha f$ are integrable on $[a,b]$ too, and

$$\int (f+g) = \int f + \int g, \quad \int \alpha f = \alpha \int f.$$

We can extend in following proposition condition of a denumerable set for Kurzweil-Henstock to a set with measure zero in case of Mcshane integral.

Proposition 2.5 Let R be a Dedekind complete Riesz space and solid ,and weakly $\sigma -$distributive.If $Q \subset [a,b]$ be a set , with $|Q| = 0$, and $f : [a,b] \to R$ be a function ,such that $f(x) = 0$ for all $x \in [a,b] \setminus Q$ Then

$$(DM) \int_a^b f = 0$$

Proof:As R is Dedekind complete it is also Archimedean space .In order to prove the proposition ,it is enough to show that there exists a $(D) -$ sequence $\left(a_{i,j}\right)_{i,j}$ on R such that for every $\varphi \in \Phi$ exists a gauge $\delta: [a,b] \to \mathbb{R}^{+}$ And

$|\sum_{\Pi} f| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$ For each $\delta -$ fine partition Π.

Let be $t_1, t_2, \ldots, t_{n,} \ldots$ a countable sequence of points from Q. By virtue of solid property , the interval $[f(t_i), f(t_j)] \in R$. Denote $\alpha = \inf\{f(t)\}$ and $\beta = \sup\{f(t)\}$. It is easy to prove that

$$R = \cup_{i,j=1}^{\infty}[f(t_i), f(t_j)] \cup [\alpha, \beta]$$

Let we consider a remuneration of indexes (i, j) of this sum .Denote with $E_n = \{t \in [a, b]: [f(t) \in f(t_{n-1}), f(t_n)]\}$.As the E_n is a subsets of Q then $|E_n| = 0$. Based on the assumption that $\mu: [0,1] \to \mathbb{R}$ is a regular for every $E \subset [a, b]$ there exists $(D) -$ sequence $(a_{i,j})_{i,j}$ such that for every $\varphi \in \Phi$ there are closed sets F_n and open sets G_n in $[a, b]$ such that

$$F_n \subset E_n \subset G_n \text{ and } \mu(G_n) \leq \mu(F_n) + \frac{1}{\beta}\bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}.$$

Define a gauge $\delta: [a, b] \to \mathbb{R}^+$ such $\delta(t) = 1$ for $t \in [a, b] \setminus Q$ and $]t - \delta(t), t + \delta(t)[\subset G_n$ if $t \in Q$ .For every δ –fine partition $\Pi = \{(J_{1,} t_1), \ldots, (J_{q,} t_q)\}$ of $[a, b]$ we have $0 \leq S(f, \Pi) = \sum_{i=1}^{q} f(t_i)|J_i| \leq \sum_{n=1}^{\infty} \sum_{i=1, t_1\in E_n}^{q} f(t_n)\, |J_i| \leq$

$$\leq \sum_{n=1}^{\infty} f(t_n) \sum_{i=1, t_1\in E_n}^{q} |J_i| \leq \sum_{i=n}^{\infty} f(t_n)|G_n| \leq \beta\frac{1}{\beta}\bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

Proposition2.6(Bolzano-Cauchy condition).A mapping $f: [a, b] \to R$ is integrable if and only if the following condition is satisfied: there exists a $(D) -$ sequence $(a_{i,j})_{i,j}$such that ,for every $\varphi \in \Phi$, there is a map $\delta: [a, b] \to \mathbb{R}^+$ so that

$$\left|\sum_{\Pi_1} f - \sum_{\Pi_2} f\right| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

whenever $\Pi_{1,}\Pi_2$ are $\delta -$fine partitions of $[a, b]$.

Proposition 2.7 If f is integrable on $[a, b]$,and $[c, b] \subset [a, b]$, then f is integrable on $[c, b]$ too .

Proposition 2.8 Let $c \in (a, b)$, $f: [a, b] \to R$ Be integrable both on $[a, c]$ and on $[c, b]$.Then f is integrable on $[a, b]$ too and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof : Since f is integrable on $[a,c]$ and on $[c,b]$,there are regulators $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$such that for any $\varphi \in \Phi$ there exist maps $\delta_1: [a,b] \to \mathbb{R}^+$, $\delta_2: [a,b] \to \mathbb{R}^+$such that

$$\left|\sum_{\Pi_1} f - \int_a^c f\right| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}, \quad \left|\sum_{\Pi_2} f - \int_c^b f\right| \leq \bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)}$$

Whenever Π_1 and Π_2 are $\delta_1-$and δ_2-fine partitions of $[a,b]$ respectively . Construct $(c_{i,j})_{i,j}$ according with Proposition 2.2 and $\delta: [a,b] \to \mathbb{R}^+$ |In the following way:

$$\delta(x) = \begin{cases} \delta_1(x) & \text{për } x < c \\ \min(\delta_1, \delta_1) & \text{për } x = c \\ \delta_2(x) & \text{për } x < c \end{cases}$$

Then to any $\delta-$fine partition Π of $[a,b]$ there are partitions Π_1 of $[a,c]$ Π_2 of $[c,b]$, $\delta_1-$and δ_2-fine , respectively , such that

.
$$\sum_{\Pi} f = \sum_{\Pi_1} f + \sum_{\Pi_2} f$$

Therefore :
$$\left|\sum_{\Pi} f - \int_a^c f - \int_c^b f\right| \leq \left|\sum_{\Pi_1} f - \int_a^c f\right| + \left|\sum_{\Pi_2} f - \int_c^b f\right| \leq$$

$$\leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)} \bigvee_{i\in 1}^{\infty} c_{i,\varphi(i)}.$$

We have obtained that f is integrable on $[a,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

3. The strong version of Mcshane Integral

Definition 3.1 We say that $f: [a,b] \to R$ has the property $\mathcal{D}(S^*M)$ on $[a,b]$ if for two $\delta-$fine partitions $\Pi_1 = \{(I_i, t_i), i = 1, \ldots, k\}$and $\Pi_2 = \{(J_j, s_j), j = 1, \ldots, m\}$ there exist a $(D)-$ sequence$(a_{i,j})_{i,j}$such that , for every $\varphi \in \Phi$ $\sum_{i=1}^{k}\sum_{j=1}^{m}|f(t_i) - f(s_j)||I_i \cap J_j| \leq \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$

It very known lemma.

Lemma 3.2 Let $\mathcal{D}= \{(I_i, t_i), i = 1, \ldots, m\}$ and $\mathcal{J}= \{(J_j, s_j)J_i, j = 1, \ldots, m\}$ be $\delta-$fine partitions of I then $\mathcal{D}' = \{(I_i \cap J_j, t_i): i = 1, \ldots, k; j = 1, \ldots, m; I_i^0 \cap J_i^0 \neq \emptyset\}$ is a $\delta-$fine partition of I and $S(f,\mathcal{D}) = S(f,\mathcal{D}')$.

From the above proposition follows that for the function $f:[a,b]\to R$ the indefinite $F(I)$ is defined as an additive $R-$valued function .We shall denote with

$$F(I) = (DM)\int_I f.$$

Boccuto &Skvortsov [2] proved a version of Henstock Lemma.

Corollary 3.3 If f is $(DM)-$integrable on $[a,b]$ and F is its indefinite integral then for every $\delta-$fine partition Π, there exist a $(D)-$sequence $\left(a_{i,j}\right)_{i,j}$ such that for every $\varphi\in\Phi$

$$\sum_{\Pi}|f(x)|I|-F(I)|\le \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

Proposition 3.4 Let R be Dedekind complete Riesz Space . A function $f:[a,b]\to R$ is (DM) integrable on $I\subset[a,b]$, if and only if,for the $\delta-$fine partitions $\mathcal{D}=\{(I_i,t_i), i=1,\dots,m\}$ and $\mathcal{E}=\{(E_j,s_j), j=1,\dots,n\}$ there exist a$(D)-$sequence $\left(a_{i,j}\right)_{i,j}$ such that,for every $\varphi\in\Phi$

holds $\sum_{i=1}^{m}\sum_{j=1}^{n}\left|f(t_i)-f(s_j)\right|\left|I_i\cap E_j\right|\le \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$

Proof . Suppose that the function f has the property D -(S^*M). Let we consider two $\delta-$fine partitions $\mathcal{D}=\{(I_i, t_i), i=1,\dots,n\}$ and $\mathcal{E}=\{(E_j,s_j), j=1,\dots,m\}$ of I.

We have $|I_i|=\sum_{j=1}^{m}\left|I_i\cap E_j\right|$ and $\left|E_j\right|=\sum_{i=1}^{n}\left|I_i\cap E_j\right|$

we observe that $\left|\sum_{i=1}^{n} f(t_i)\,|I_i|-\sum_{j=1}^{m} f(s_j)\,\left|E_j\right|\right|.$

$$=\left|\sum_{j=1}^{m}\sum_{i=1}^{n} f(t_i)\left|I_i\cap E_j\right|-\sum_{i=1}^{n}\sum_{j=1}^{m} f(s_j)\left|I_i\cap E_j\right|\right|\le$$

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(t_i)-f(s_j)\left|I_i\cap E_j\right|\right|\le \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

Since f has the property $\mathcal{D}$ -(S^*M) we have proved the conditions of necessary.For the converse let we set $\mathcal{F}=\{K_{i,j}=I_i\cap E_j, I_i, E_j$ are respetively elements of partitions $\mathcal{D}_1$ and $\mathcal{D}_2$, $\mathring{K}_{i,j}\cap\mathring{K}_{i',j'}=\varphi\}$.We know that $\left|f(t_i)-f(s_j)\right|=f(t_i)\vee f(s_j)-f(t_i)\wedge f(s_j)$.Define the tags t_{ij} and s_{ij} as follows $f(t_{ij})= f(t_i)\vee f(s_j)$ and $f(s_{ij})= f(t_i)\wedge f(s_j)$.

We get $\left|f(t_i)-f(s_j)\right|=f(t_{ij})-f(s_{ij})$.

Let $\mathcal{D}' = \{(K_{i,j}; t_{ij}) : K_{i,j} \in \mathcal{F}\}$ and $E' = \{(K_{i,j}; s_{ij}) : K_{i,j} \in \mathcal{F}\}$ be two partitions .By the Henstock Lemma $\mathcal{D}'$ and E' are δ −fine partitions of I , so by hypothesis we have

$$|S(f,\mathcal{D}') - S(f,E')| \le \left|S(f,\mathcal{D}') - (DM)\int_a^b f\right| + \left|S(f,E') - (DM)\int_a^b f\right| \le \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

On the other hand

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left|f(t_i) - f(s_j)\right|\left|I_i \cap E_j\right| = \left|\sum_{K_{i,j}}\left[f(t_{ij}) - f(s_{ij})\right]\left|I_i \cap E_j\right|\right| =$$

$= |S(f,\mathcal{D}') - S(f,E')|$ .This proves the theorem.

A net $(p_\beta)_{\beta\epsilon\Lambda}$ in $\widetilde{\mathbb{R}}$ where$(\Lambda, \ge) \neq \emptyset$ is directed set, is called (o)-net if is decreasing (i. e. $p_{\beta_1} \le p_{\beta_1}$ whenever $\beta_1, \beta_1 \epsilon \Lambda$ $\beta_1 \ge \beta_2$) and $\inf_{\beta p \beta}$ =0.In particular we get the definition of sequence (o) −convergence when $\Lambda = N$

Definition 3.5 We say that a net $(r_\beta)_\beta$ order converges (or short (o) −converges) to $r\epsilon R$ if there exists an (o) − net $(p_\beta)_{\beta\epsilon\Lambda}$ satisfying $|r_\beta - r| \le p_\beta$ for each $\beta\epsilon\Lambda$.

Proposition 3.6 Let R be a Dedekind complete regular Riesz Space and $f \ge 0$ be $D(S^*M)$ −integrable on a solid $E \subset [a,b]$.Then there exist the function g and h DM −integrable such that $0 \le g \le f \le h$

and there exist a directed net $(p_\delta)_{\delta\epsilon\Delta}$ such that $(DM)\int_E |f - g| \le p_\delta$

Proof .Construct the simple function

$$f(x) = \begin{cases} \dfrac{k-1}{2^n}u & \text{if} \quad \dfrac{k-1}{2^n}u \le f(x) \le \dfrac{k}{2^k}u \\ 0 & \text{if contrary} \end{cases}$$

Where $k = 1,2, \dots, n2^n$ and u unit element of R .We get that

$$\left|f(x) - f_{n(x)}\right| \le \frac{1}{2^n}u \qquad (1)$$

We get that sequence f_n is (r) −convergent to f(x) .If we write functions in a form $f_n(x) = \sum_{i=1}^{n} k_i \chi_{E_i}$. We get that there S_n and positive elements $c_n \epsilon R$ such that $f(x) = \sum_{n=1}^{\infty} c_n \chi_{S_n}$ for every $x\epsilon E$ .Moreover f is integrable and

$$\sum_{n=1}^{\infty} c_n |S_n| = \int_E f < +\infty \quad (2)$$

Since S_n are Lebegue measurable there exist compact sets K_n and open sets G_n of $[a, b] \cap E$ that $K_n \subset S_n \subset G_n$and there exists $\varepsilon > 0$ such that $|G_n \backslash K_n| < \epsilon$.

From the equation (1) we take $c_n |G_n \backslash K_n| < \frac{u\epsilon}{2^{n+2}}$. From the convergence of the series (2) we get $\sum_{N+1}^{\infty} c_n |S_n| < \frac{\varepsilon U}{4}$ Define $g = \sum_{i=1}^{N} c_n \chi_{k_n}$ and $h = \sum_{i=1}^{N} c_n \chi_{G_n}$. It is easy to note that $g \leq f \leq h$ and

$$h - g = \sum_{i=1}^{N} c_n \chi_{G_n \backslash K_n} + \sum_{i=N+1}^{\infty} c_n \chi_{G_n} \leq \sum_{i=1}^{\infty} c_n \chi_{G_n \backslash K_n} + \sum_{i=N+1}^{\infty} c_n \chi_{S_n}$$

By virtue of proposition 2.5 we proved the proposition.

4. The fundamental theorems of Calculus for the DM-integral

A function f is said to be continuous at a point $x_0 \ \epsilon [a, b]$ if there exists $(D)-$ sequence $(a_{i,j})_{i,j}$in R such that, to any $\varphi \in \Phi$ there exist $f : [a, b] \to \mathbb{R}^+$such that $|f(x) - f(x_0)| \leq \bigvee_{i \in 1}^{\infty} a_{i,\varphi(i)}$ for $\forall \ (I, x) \epsilon B(x_0, \delta)$.

Given $E \neq \emptyset$ and $E \subset [a, b]$ we say that the function f is -continuous on E if it is continuous at every point of E. We say that f is $(u)-$ differentiable on E if there exists a function g: E→ R such that

$$\left|\frac{f(I)}{|I|} - g(x)\right| \leq \bigvee_{i \in 1}^{\infty} a_{i,\varphi(i)} \quad \text{for} \ \forall \ (I, x) \epsilon B(x_0, \delta).$$

The function g is called the $(u)-$ derivative with respect to f. It is easy to prove that $(u)-$ derivative is determined uniquely.

Theorem 4.1 Let R be a Dedekind complete Riezs space , f be a R−valued function $(u)-$ differentiable with on $[a, b]$ with derivative f',then f' is DM −integrable on $[a, b]$,and

$$\int_a^b f' = f([a, b]).$$

Proof By $(u)-$ differentiability of f in $[a, b]$, there exists $a(D)-$sequence $(a_{i,j})_{i,j}$in R such that , to any $\varphi \in \Phi$ and gauge $\delta: [a, b] \to \mathbb{R}^+$ such

$$\left|\frac{f(I)}{|I|} - g(x)\right| \leq \bigvee_{i \in 1}^{\infty} a_{i,\varphi(i)} \quad \text{for every } (I, x) \in B(x_0, \delta).$$

Choose a δ −fine partition P= $\{(I_i x_i): i = 1, \dots, q\}$ of $[a, b]$.

From the above inequality, we get

$$0 \leq |\sum(f'P) - f[a,b]| = \left|\sum_{i=1}^{q} f'(x_i)|I_i| - f(I_i)\right| \leq \sum_{i=1}^{q}\left\{|I_i|\left|\frac{f(I_i)}{I_i} - f'(x_i)\right|\right\} \leq \left(\sum_{i=1}^{q}|I_i|\right)\bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)} = (b-a)\bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)}$$

Let we follow the idea of [4] for the function Mcshane integrable with value real to prove the theorem:

Theorem 4.2 Let R be a regular Riesz space, $f:[a,b] \to R$ and let F be a R−valued function, such that for some set $Q \subset [a,b]$ with $|Q| = 0$.If the function f is $(u)-$ derivative of F on $[a,b]\backslash Q$. Then f is Mcshane integrable in $[a,b]$ and

$$(DM)\int_a^b f = (F[a,b]).$$

Proof Since the Mcshane integrability by virtue of Proposition 2.7 does not depend on values of f on a set of f measure zero, we may assume $f(x) = 0$ on Q. Let be $K = [a,b]\backslash Q$. As f is the $(u)-$ derivative of F in K, exist a $(D)-$ sequence $(a_{i,j})_{i,j}$ in R such that, to any $\varphi \in \Phi$ there exist gauge $\delta_1: [a,b] \to \mathbb{R}^+$. It can founded such that if $P = \{(I_i, x_i): i = 1,2, \dots, q\}$ decomposition, then

$$\left||I_i|f(x_i) - F(|I_i)\right| \leq |I_i| \bigvee_{i\in 1}^{\infty} a_{i,\varphi(i)} \text{ For all i= } 1, \dots, q \text{(see [1,proposition 2.22])}$$

For the interval within of Q we choose some intervals such that $\sum_{j=1}^{n}|I_i| < \varepsilon$, where $I_1, I_2, \dots, I_n$ are non overlapping intervals .Moreover by continuity of F exist a $(D)-$ sequence $(b_{i,j})_{i,j}$ in R such that, to any $\varphi \in \Phi$ there exist $\delta: [a,b] \to \mathbb{R}^+$ such that

$$\sup\{|F([u,v]): x - \delta \leq u \leq x \leq v \leq \delta|\} \leq \bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)}$$

We observe $\sum_{j=1}^{n}\left|F(I_j)\right| < \frac{1}{n}\bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)}$ for $I_j \subset Q$ there is gauge δ_2 such that

$$\sum_{(I,x)\in P, x\in E}|F(I)| < \bigvee_{i\in 1}^{\infty} b_{i,\varphi(i)}$$

Put $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$.Then for every δ −fine partition $P = \{(I_i, x_i): i = 1,2, \dots, q\}$ of $[a,b]$ we have

$0 \leq \left|\left[\sum_{i=1}^{q} |I_i| f(x_i)\right] - F([a,b])\right| = \left|\sum_{i=1}^{q} \{|I_i| f(x_i) - F(I_i)\}\right| \leq \left|\sum_{x_i \notin Q} \{|I_i| f(x_i) - F(I_i)\}\right| + \sum_{x \in Q} |F(I_i)| \leq \bigvee_{i \in 1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i \in 1}^{\infty} b_{i,\varphi(i)}$.

From this the assertion follows.

5. Applications to Walsh Series

In this section we consider Walsh series with coefficients from a Riesz space and give an application of theorem (4.2) Coefficients of a convergent Walsh series can determinate from its sum by generalized Furier formulas .We can use the Integral of Mcshane for Riesz-space-valued functions . By the proposition 2.22 to article[1] related to weak σ-distributivy in l-group we can formulate:

Lema 5.1 [2] Let $(a_n)_n$ be a sequence (D) −convergent to zero in a Dedekind complete Riesz space R .Then the sequence $(\sigma_j)_j$ of its arithmetical means also (D) −converges to zero.

We present shortly following the Walsh functions using dyadic expansions of natural numbers as well as those of real number of the half –open interval $[0,1)$. Let $n = \sum_{j=o}^{\infty} \varepsilon_j 2^j$ with $\varepsilon_j = 0$ or 1, and $x = \sum_{j=o}^{\infty} x_j 2^{-j-1}$ with $x_j = 0$ or 1 .For the dyadic rationals x we use only finite expansions .We put

$$w_n(x) = (-1)^{\sum_{j=o}^{\infty} \varepsilon_j x_j} \quad n \epsilon N, \ x \epsilon [0,1).$$

Note that for $n \leq 2^k$ the functions w_n are constant on each interval Δ_i^k, where $\Delta_i^k = \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right)$, $k \epsilon N$, $i = 0,1, \dots, 2^k - 1$.

Let $S_n = \sum_{j=0}^{n-1} a_j w_j$ be the partial sum of a Walsh series $\sum_{j=0}^{\infty} a_j w_j$ (3)

With coefficients a_j belonging to a Dedekind complete Riesz space R which is wekly σ-distributive. The sums S_n are constant for $n < 2^k$ on each interval Δ_i^k

In context of weakly σ-distributive R we consider pointwise (D) − convergence (order convergence) well as the following (u) − convergence on a set.

Definition 5.2 Let Λ be any nonempty set, R be any (arbitrary) Dedekind complete Riesz space and D= N^{Λ} .We say that the sequence of R-valued functions $(S_n(x))_n$, $x \epsilon \Lambda$, (u) − converges to the function $S: \Lambda \to R$ if exists an (D) − sequence $(p_v)_{v \in D}$ that ∀ $\varphi \in \phi$, there exist $v(x) \in D$ such that $n > v(x)$

$|S_n(x) - S(x)| \leq \bigvee_{i \in 1}^{\infty} a_{i,\varphi(i)}$ and $x \in \Lambda$.

The crucial step of the solution of the coefficients problem for the Walsh series is to observe that the integral (DM) $\int_{D_i^k} S_{2^k}$ defines an additive interval function ψ on the family D

Lemma5. 3 [2]

That $\psi(D_i^k) = \psi(D_{2i}^{k+1}) + \psi(D_{2i+1}^{k+1})$ as the sum S_{2^k} is constant on each D_i^k then $S_{2^k}(x) = \frac{\psi(D_i^k)}{|D_i^k|}$, where $x \in D_i^k$. (4)

It follows from this formula that if the Walsh series is $(u) -$ convergent on the same set of dyadic –irrational point , then the function ψ is $(u) -$ differentiable on the same set.

Proposition 5.4

If the coefficients of a Walsh series form an $(D) -$ sequence convergent to zero , then the corresponding function ψ is $(D) -$ continuous at each point of $[0,1]$.

We now prove the following theorem on recovering the coefficients of a Walsh series from its sum, extended in case of Mcshane integration.

Theorem 5.5 If R is a regular Riesz space and a Walsh series (3)is $(u) -$ convergent to a function f on a set $[0,1[\setminus E$,where E is subset of $[0,1[$, with $\mu(E) = 0$ then f is Mcshane integrable on $[0,1]$ and the series (3) is the Fourier series of f in the sense of the Mcshane integral .

Proof In our context, series being $(u) -$convergent is also $(D) -$convergent on $[0,1[\setminus E$.Note that the $(D) -$ convergence of Walsh series at least at one point implies that coefficients of series $(D) -$ converge to zero. Then the function ψ defined for our series according to proposition 5.3 is $(D) -$ continuous everywhere on $[0.1]$ .Denote by Q the set of dyadic –rational points .It follows from the definitions of $(u) -$ convergence , $(u) -$ diffentiability and from equality (4) that the function ψ is $(u) -$ diffentiable with $(u) -$ derivative f in $[0,1[\setminus(E \cup Q)$.In these conditions we can apply to functions ψ and f Theorem 4.2 to get

$$(DM) \int_{\alpha}^{\beta} f = \psi([\alpha, \beta])$$

where $[\alpha, \beta]$ is dyadic interval D_i^k.

Note that for $n < 2^k$ the coefficients a_n are the Fourier coefficients of the partial sumS_{2^k}. Then, denoting by w_{ni} the value of the function w_n on D_i^k we get

$$a_n = \int_0^1 S_{2^k} w_n = \sum_{i=0}^{2^k-1} \int_{D_i^k.} S_{2^k} w_n = \sum_{i=0}^{2^k-1} w_{ni} \int_{D_i^k.} S_{2^k} =$$

$\sum_{i=0}^{2^k-1} w_{ni}\, \psi\left(D_i^k\right) = \sum_{i=0}^{2^k-1} w_{ni}\, (DM) \int_{D_i^k.} f = (DM) \int_0^1 fw_n$ .This completes the proof.

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