Endpoint and a Common Fixed Point for Multi-Valued Mappings

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Abstract

In this paper, we study endpoint and a common fixed points for some generalized multi-valued nonexpansive mappings which is a multi-valued version of condition $(C_\lambda)$ defined by Garcia-Falset et al.[3].

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1 Introduction

A mapping $T$ on a subset $E$ of a Banach space $X$ is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. We denote by $F(T)$ the set

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of fixed points of $T$, i.e., $F(T) = \{ x \in E : Tx = x \}$. A mapping $T : E \to E$ is called a quasi nonexpansive if $F(T) \neq \emptyset$ and $\| Tx - z \| \leq \| x - z \|$ for all $x \in E$ and $z \in F(T)$. Suzuki [7] introduced a condition $(C)$ and proved fixed point theorems and convergence results for mappings satisfying the condition $(C)$. Moreover, he presented that the condition $(C)$ is weaker than nonexpansiveness and stronger than quasinonexpansiveness.

**Definition 1.1.** Let $T$ be a mapping on a subset $E$ of a Banach space $X$. Then $T$ is said to satisfy condition $(C)$ if for each $x, y \in E$,

\[
\frac{1}{2} \| x - Tx \| \leq \| x - y \| \quad \text{implies} \quad \| Tx - Ty \| \leq \| x - y \|.
\]

Garcia-Falset et al. [3] presented a generalization of condition $(C)$ as follows.

**Definition 1.2.** For $\lambda \in (0, 1)$, we say that a mapping $T : E \to X$ satisfy condition $(C_\lambda)$ on $E$ if for all $x, y \in E$ with

\[
\lambda \| x - Tx \| \leq \| x - y \| \quad \text{implies} \quad \| Tx - Ty \| \leq \| x - y \|.
\]

Of course, if $\lambda = \frac{1}{2}$ we recapture the class of mappings satisfying condition $(C)$. Notice that if $0 < \lambda_1 < \lambda_2$ then the condition $(C_{\lambda_1})$ implies condition $(C_{\lambda_2})$.

In this paper, we study endpoint theorems and a common fixed point for some generalized multi-valued nonexpansive mappings which is a multi-valued version of condition $(C_\lambda)$.

## 2 Preliminary

Let $X$ be a Banach space and $E$ a nonempty subset of $X$. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of $E$, by $F(E)$ the family of nonempty closed subsets of $E$, by $K(E)$ the family of nonempty compact subsets of $E$, by $FC(E)$ the family of nonempty closed convex subsets of $E$, and by $KC(E)$ the family of nonempty compact convex subsets of $E$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

\[
H(A, B) = \max \{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \}, \quad A, B \in FB(X),
\]

where $D(a, B) = \inf \{ \| a - b \| : b \in B \}$ is the distance from the point $a$ to the subset $B$. A multi-valued mapping $T : E \to FB(X)$ is said to be nonexpansive if $H(Tx, Ty) \leq \| x - y \|$ for all $x, y \in E$. We denote by $F(T)$ the set of fixed
points of $T$, i.e., $F(T) = \{x \in E : x \in Tx\}$. An element $x \in E$ said to be an endpoint of $T$, if $Tx = \{x\}$.

Kaewcharoen and Panyanak [4] introduced a condition on multi-valued mappings which was a multi-valued version of condition $(C_\lambda)$ defined by Garcia-Falset et al. [3].

**Definition 2.1.** Let $T : E \to FB(X)$ be a multi-valued mapping. Then $T$ is said to satisfy condition $(C_\lambda)$ for some $\lambda \in (0, 1)$ if for each $x, y \in E$,

$$\lambda D(x, Tx) \leq \|x - y\| \quad \text{implies} \quad H(Tx, Ty) \leq \|x - y\|.$$ 

A multi-valued mapping $T : X \to FB(X)$ is said to be a weak contraction if there exists $0 \leq \alpha < 1$ such that

$$H(Tx, Ty) \leq \alpha N(x, y)$$

for all $x, y \in X$, where

$$N(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}.$$ 

Nadler [6] extended the Banach contraction principle to multi-valued mapping as follows.

**Theorem 2.2.** Let $(X, d)$ be a complete metric space. Suppose that $T : X \to FB(X)$ is a contraction mapping in the sense that for some $0 \leq \alpha < 1$,

$$H(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$.

Daffer and Kaneko [2] proved the existence of a fixed point for a multi-valued weak contraction mappings of a complete metric space $X$ into $FB(X)$ as follows.

**Theorem 2.3.** Let $(X, d)$ be a complete metric space. Suppose that $T : X \to FB(X)$ is a contraction mapping in the sense that for some $0 \leq \alpha < 1$,

$$H(Tx, Ty) \leq \alpha N(x, y)$$

for all $x, y \in X$, where

$$N(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}.$$ 

If $x \mapsto d(x, Tx)$ is lower semicontinuous, then there exists a point $x_0 \in X$ such that $x_0 \in Tx_0$. 

In the following theorem, Amini-Harandi [1] proved endpoint result for a multi-valued mappings of a complete metric space $X$ into $FB(X)$.

**Theorem 2.4.** Let $(X,d)$ be a complete metric space. Suppose that $T : X \rightarrow FB(X)$ is a multi-valued mapping that satisfies
\[
H(Tx,Ty) \leq \psi(d(x,y)),
\]
for each $x, y \in X$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is upper semicontinuous, with $\psi(t) < t$ for all $t > 0$, satisfying $\liminf_{t \rightarrow \infty} (t - \psi(t)) > 0$. Then $T$ has a unique endpoint if and only if $T$ has the approximate endpoint property.

In the following theorem, Moradi and Khojasteh [5] extended result of Nadler [6], Daffer and Kaneko [2], and Amini-Harandi [1].

**Theorem 2.5.** Let $(X,d)$ be a complete metric space. Suppose that $T : X \rightarrow FB(X)$ is a multi-valued mapping that satisfies
\[
H(Tx,Ty) \leq \psi(N(x,y)),
\]
for each $x, y \in X$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is upper semicontinuous, with $\psi(t) < t$ for all $t > 0$, satisfying $\liminf_{t \rightarrow \infty} (t - \psi(t)) > 0$. Then $T$ has a unique endpoint if and only if $T$ has the approximate endpoint property.

**Lemma 2.6.** ([3],[7]) Let $(E,d)$ be a complete metric space, $A, B \in FB(E)$ and $a \in A$. Then for each $\varepsilon > 0$, there exists $b \in B$ such that
\[
d(a,b) \leq H(A,B) + \varepsilon.
\]

**Lemma 2.7.** ([2]) Let $E$ be a nonempty bounded convex subset of a Banach space $X$. Let $T : E \rightarrow FB(E)$ be a multi-valued mapping satisfying condition $(C_\lambda)$ for some $\lambda \in (0,1)$. If $\{x_n\}$ is the sequence defined by
\[
x_{n+1} = (1 - \lambda)x_n + \lambda y_n
\]
where $y_n \in Tx_n$, then $\lim_{n \rightarrow \infty} D(x_n,Tx_n) = 0$.

## 3 Main Results

In this section, we prove our main results.

**Theorem 3.1.** Let $E$ be a nonempty bounded convex subset of a Banach space $X$. Let $T : E \rightarrow FB(E)$ and $S : E \rightarrow FB(E)$ be a multi-valued mapping satisfying
\[
H(Tx,Sy) \leq \psi(N(x,y)),
\]
where

\[ N(x, y) = \max \{ \|x - y\|, D(x, Tx), D(y, Sy), \frac{D(x, Sy) + D(y, Tx)}{2} \}, \]

for each \( x, y \in E \), \( \psi : [0, +\infty) \to [0, +\infty) \) is u.s.c. with \( \psi(t) < t \) for all \( t > 0 \) and \( \lim\inf_{t \to +\infty} (t - \psi(t)) > 0 \). If \( T \) and \( S \) satisfying condition \((C_\lambda)\) for some \( \lambda \in (0, 1) \), then \( T \) and \( S \) have a unique endpoint. Moreover, \( T \) and \( S \) have a unique common fixed point.

**Proof.** Defined a sequence \( \{x_n\} \in E \) by \( x_0 \in E \) and choose \( y_0 \in Tx_0 \). Let

\[ x_1 = (1 - \lambda)x_0 + \lambda y_0. \]

By Lemma 2.6, there exists \( y_1 \in T x_1 \) such that \( \|y_1 - y_0\| \leq H(Tx_0, Tx_1) + \gamma_0 \).

Let

\[ x_2 = (1 - \lambda)x_1 + \lambda y_1. \]

Therefore for every natural number \( n \geq 1 \), we have

\[ x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \]

where \( y_n \in Tx_n \) such that

\[ \|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n. \]

By Lemma 2.7, we have

\[ \lim_{n \to \infty} D(x_n, Tx_n) = 0. \]

Thus

\[ \lim_{n \to \infty} H(\{x_n\}, Tx_n) = 0. \]

Similarly,

\[ \lim_{n \to \infty} H(\{x_m\}, Sx_m) = 0. \]
For all \( m, n \in \mathbb{N} \), we have
\[
N(x_n, x_m) = \max\{ \|x_n - x_m\|, D(x_n, Tx_n), D(x_m, Sx_m), \frac{D(x_n, Sx_m) + D(x_m, Tx_n)}{2} \} \\
\leq \max\{ \|x_n - x_m\|, H(\{x_n\}, Tx_n), H(\{x_m\}, Sx_m), \frac{H(\{x_n\}, Sx_m) + H(\{x_m\}, Tx_n)}{2} \} \\
\leq \|x_n - x_m\| + H(\{x_n\}, Tx_n) + H(\{x_m\}, Sx_m) \\
= \|x_n - x_m\| - H(\{x_n\}, Tx_n) - H(\{x_m\}, Sx_m) \\
+ 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Sx_m) \\
\leq H(Tx_n, Sx_m) + 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Sx_m) \\
\leq \psi(N(x_n, x_m)) + 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Sx_m).
\]

Since \( \psi \) is u.s.c., \( \psi(t) < t \) for all \( t > 0 \) and \( \lim_{n \to \infty} \frac{1}{n} \psi(t - \psi(t)) > 0 \), we have \( \limsup_{n \to \infty} N(x_n, x_m) = 0 \). Thus \( \{x_n\} \) is a Cauchy sequence. So there exists \( x_0 \in E \) such that \( \lim_{n \to \infty} x_n = x_0 \). Note that if \( N(x_n, x_0) = 0 \) for some \( n_0 \in \mathbb{N} \), then \( D(x_0, Tx_0) = 0 \). This means that \( x_0 \in \overline{Tx_0} = Tx_0 \). Similarly, we have \( x_0 \in \overline{Sx_0} = Sx_0 \). Thus \( x_0 \in Tx_0 \) and \( x_0 \in Sx_0 \). Suppose that \( N(x_n, x_0) \neq 0 \) for all \( n \in \mathbb{N} \). Then we have
\[
H(\{x_n\}, Tx_0) - H(\{x_n\}, Sx_n) \leq H(Sx_n, Tx_0) \\
\leq \psi(N(x_n, x_0)) \\
< N(x_n, x_0) \\
\leq \|x_n - x_0\| + H(\{x_n\}, Sx_n) \\
+ H(\{x_0\}, Tx_0).
\]

This shows that \( \lim_{n \to \infty} N(x_n, x_0) = H(\{x_0\}, Tx_0) \). Since \( \psi \) is u.s.c., we have
\[
\limsup_{n \to \infty} \psi(N(x_n, x_0)) \leq \psi(H(\{x_0\}, Tx_0)).
\]

By (1) and (2), we conclude that
\[
H(\{x_0\}, Tx_0) \leq \psi(H(\{x_0\}, Tx_0)).
\]

Hence \( H(\{x_0\}, Tx_0) = 0 \). This means that \( Tx_0 = \{x_0\} \). Similarly, we have \( Sx_0 = \{x_0\} \). It follow that \( x_0 \in Tx_0 = Sx_0 \). Thus \( T \) and \( S \) have an endpoint.
Moreover, $T$ and $S$ have a common fixed point. To prove the uniqueness of the endpoint of $T$, let $x$ be an arbitrary endpoint of $S$. Then $Sx = \{x\}$. Consider,

\[
\begin{align*}
\|x_0 - x\| &= D(x_0, Sx) \\
&\leq H(Tx_0, Sx) \\
&\leq \psi(N(x_0, x)) \\
&< N(x_0, x) \\
&= \max\{\|x_0 - x\|, D(x_0, Tx_0), D(x, Sx), \\
&\quad D(x_0, Sx) + D(x, Tx_0)\} \\
&\leq \max\{\|x_0 - x\|, H(\{x_0\}, Tx_0), H(\{x\}, Sx), \\
&\quad H(\{x_0\}, Sx) + H(\{x\}, Tx_0)\} \\
&= \max\{\|x_0 - x\|, H(\{x_0\}, \{x_0\}), H(\{x\}, \{x\}), \\
&\quad H(\{x_0\}, \{x\}) + H(\{x\}, \{x_0\})\} \\
&= \|x_0 - x\|.
\end{align*}
\]

We obtain $\|x_0 - x\| < \|x_0 - x\|$, and this is a contradiction. It follow that $\|x_0 - x\| = 0$. Therefore $x_0 = x$ and this completes the proof.

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