On the Rate Space of Analytic Sequences

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Abstract

Let $\Lambda_\pi$ denote the space of all analytic rate sequences.
Let $\Gamma_\pi$ denote the space of all entire rate sequences.
A subset of $\Gamma_\pi$ is $\chi_\pi$. This paper is devoted to a study of general properties of $\Lambda_\pi$, $\Gamma_\pi$, $\chi_\pi$.

We note that $\chi_\pi \subset \Gamma_\pi \subset \Lambda_\pi$

Mathematics Subject classification: 46A45

Keywords: Entire Rate sequences, Analytic Rate sequences, Norlund space, Quasi-complete, Barreled space, semi reflexive spaces
INTRODUCTION

Based on Norlund transformation, we have formulated the Norlund space of $\Gamma$ and the Norlund space of $\Lambda$ and the Norlund space of $\chi$.

In this section the Norlund space of entire rate sequences $\eta$ $(\Gamma_\pi)$ is introduced and it is shown that $\eta$ $(\Gamma_\pi) = \Gamma_\pi$ the space of entire rate sequences.

We have also examined as to whether the space $\Gamma_\pi$ is quasi complete space, barrelled space and semi-reflexive space.

Definition.1 A complex sequence whose $k^{th}$ term is $x_k$ will be denoted by $(x_k)$ or $x$. Let $\pi = \{\pi_k\}$ be a sequence of positive numbers.

A sequence $x = (x_k)$ is said to be analytic rate sequence, if

$$\left(\sum_{k=1}^{\infty} x_k^k \frac{1}{\pi_k^k}\right)^{1/k} < \infty.$$  

The vector space of all analytic rate sequences will be denoted by $\Lambda_\pi$.

Definition.2 A sequence $x$ is called entire rate sequence if

$$\lim_{k \to \infty} \frac{x_k^{k}}{\pi_k^k} = 0$$  

The vector space of all entire rate sequences will be denoted by $\Gamma_\pi$.

i.e., $\Gamma_\pi = \left\{ x = (x_k) : \lim_{k \to \infty} \left(\sum_{k=1}^{\infty} x_k^k \pi_k^k\right)^{1/k} = 0 \right\}$

Definition.3 A sequence $x$ is called kamthan rate sequence if

$$\left(\frac{k! x_k}{\pi_k^k}\right)^{1/k} \to 0 \text{ as } k \to \infty.$$  

The vector space of all kamthan rate sequences will be denoted by $\chi_\pi$.

i.e., $\chi_\pi = \left\{ x = (x_k) : \left(\frac{k! x_k}{\pi_k^k}\right)^{1/k} \to 0 \text{ as } k \to \infty \right\}$

Then $\chi_\pi$ is a metric space with metric
\[ d(x,y) = \sup_{(k)} \left\{ k! \left| \frac{x_k}{\pi_k} - \frac{y_k}{\pi_k} \right|^{1/k} : k = 1,2,3,\ldots \right\}. \]

**Definition. 4** The space \( \chi_\pi(\Delta) \) is defined as set of all those rate sequences \((x_k)\) such that \( (\Delta x_k) \in \chi_\pi \) where \( \Delta x_k = \frac{x_k}{\pi_k} - \frac{x_{k+1}}{\pi_{k+1}} \) for \( k = 1,2,3,\ldots \). Note that \( x = (x_k) \in \chi_\pi(\Delta) \iff \Delta x_k \in \chi_\pi \). Then \( \chi_\pi(\Delta) \) is a metric space with the metric
\[ d(x,y) = \sup_{(k)} \left\{ k! \left| \frac{\Delta x_k}{\pi_k} - \frac{\Delta y_k}{\pi_k} \right|^{1/k} : k = 1,2,3,\ldots \right\}. \]

**Definition. 5** If \( X \) is a sequence space, we define the \( \beta \)–dual \( X^\beta \) of \( X \) by
\[ X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for every } x \in X \}. \]

**Definition. 6** Let \( (p_n)_{n=0}^{\infty} \) be a sequence of nonnegative real numbers with \( p_0 > 0 \) consider
\[ y_k = \frac{p_0 x_k + p_1 x_{k-1} + \cdots + p_k x_0}{p_0 + p_1 + \cdots + p_k} \quad \text{for } k = 0,1,2,\ldots. \]
Then \( y = \{y_k\} \) is called Norlund transform of the sequence \( x = \{x_k\} \). \( \eta(\Gamma) = \{ x = (x_k) : (y_k) \in \Gamma \} \) and \( \eta(\Lambda) = \{ x = (x_k) : (y_k) \in \Lambda \} \)

The Norlund rate space of \( \Gamma_\pi \) is denoted by \( \eta(\Gamma_\pi) \) and the Norlund rate space of \( \Lambda_\pi \) is denoted by \( \eta(\Lambda_\pi) \) and defined as
\[ \eta(\Gamma_\pi) = \left\{ x = (x_k) : \left\{ \frac{x_k}{\pi_k} \right\} \in \eta(\Gamma) \right\} \]
& \[ \eta(\Lambda_\pi) = \left\{ x = (x_k) : \left\{ \frac{x_k}{\pi_k} \right\} \in \eta(\Lambda) \right\} \]

We write \( P_n = p_0 + p_1 + \ldots + p_n \) for \( n = 0,1,2,\ldots \). \n
**Definition. 7** A locally convex topological vector space \( X \) is said to be quasicomplete if each bounded set is complete in \( X \). \n
**Definition. 8** An absolutely convex absorbent closed subset of locally convex topological vector space \( X \) is called a barrelled space if each barrel is a neighborhood of zero. \n
**Definition. 9** A locally convex topological vector space \( X \) is said to be semi-reflexive if each bounded closed set in \( X \) is \( \sigma(X,X') \)– compact.
Theorem 1 \[ \eta(\Gamma_{\pi}) = \Gamma_{\pi} \]

Proof:
Let \( x \in \eta(\Gamma_{\pi}) \). Then \( y \in \Gamma_{\pi} \) so that for every \( \varepsilon > 0 \), we have a positive integer \( n_0 \) such that
\[
\left| \frac{p_0 \frac{x_n}{\pi_n} + p_1 \frac{x_{n-1}}{\pi_{n-1}} + \ldots + p_n \frac{x_0}{\pi_0}}{p_n} \right| < \varepsilon^n \quad \text{for all } n \geq n_0.
\]
Take \( p_0 = 1, p_1 = p_2 = \ldots = p_n = 0 \)
We have then \( \left| \frac{x_n}{\pi_n} \right| < \varepsilon^n \) for all \( n \geq n_0 \)
Hence \( x \in \Gamma_{\pi} \). Arbitrariness of \( x \) in \( \eta(\Gamma_{\pi}) \) gives
\[ \eta(\Gamma_{\pi}) \subset \Gamma_{\pi}. \quad \ldots \ldots (1) \]
On the other hand, let \( x \in \Gamma_{\pi} \). But then given any \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that
\[
\left| \frac{x_n}{\pi_n} \right| < \varepsilon^n \quad \text{for all } n \geq n_0 \quad \text{we have} \quad \left| \frac{y_n}{\pi_n} \right| < \frac{p_0 \varepsilon^n + p_1 \varepsilon^{n-1} + \ldots + p_n \varepsilon^0}{p_n}
\]
Hence \( y \in \Gamma_{\pi} \).
Consequently \( x \in \eta(\Gamma_{\pi}) \). Arbitrariness of \( x \) in \( \Gamma_{\pi} \) gives
\[ \Gamma_{\pi} \subset \eta(\Gamma_{\pi}) \quad \ldots \ldots (2) \]
From (1) and (2)
\[ \text{Thus } \eta(\Gamma_{\pi}) = \Gamma_{\pi} \]

Theorem 2 \[ \eta(\Lambda_{\pi}) = \Lambda_{\pi} \]

Proof:
Let \( x \in \Lambda_{\pi} \). Then there exists a positive constant \( M \) such that
\[
\left| \frac{x_n}{\pi_n} \right| \leq M^n \quad \text{for } n = 0,1,2, \ldots
\]
\[
\left| \frac{y_n}{\pi_n} \right| \leq \frac{p_0 M^n + p_1 M^{n-1} + \ldots + p_n}{p_n} \leq \frac{M^n}{p_n} \left( p_0 + \frac{p_1}{M} + \ldots + \frac{p_n}{M^n} \right) \leq \frac{M^n}{p_n} \left( p_0 + p_1 + \ldots + p_n \right) \leq M^n \quad \text{for } n = 0,1,2, \ldots
\]
Hence \( y \in \Lambda_\pi \). But then \( x \in \eta(\Lambda_\pi) \).
Consequently \( \Lambda_\pi \subseteq \eta(\Lambda_\pi) \). . . . (3)
On the other hand let \( x \in \eta(\Lambda_\pi) \). Then \( y \in \Lambda_\pi \).
Hence there exists a positive constant \( M \) such that
\[
\left| \frac{y_n}{\pi_n} \right| \leq M^n \text{ for } n = 0, 1, 2, \ldots
\]
\[
\left| \frac{p_0 x_n}{\pi_n} + \frac{p_1 x_{n-1}}{\pi_{n-1}} + \ldots + \frac{p_n x_0}{\pi_0} \right| \leq M^n p_n
\]
Take \( p_0 = 1, p_1 = p_2 = \ldots = p_n = 0 \)
Then it follows that \( P_n = 1 \) and so \( \left| \frac{x_n}{\pi_n} \right| \leq M^n \) for all \( n \)
Consequently \( x \in \Lambda_\pi \). Arbitrariness of \( x \) in \( \eta(\Lambda_\pi) \) gives
\( \eta(\Lambda_\pi) \subseteq \Lambda_\pi \). . . . (4)
From (3) and (4)
Thus \( \eta(\Lambda_\pi) = \Lambda_\pi \)

**Theorem.3** \( \Gamma_\pi \) is quasi complete.

**Proof:**
Consider the sequence \( \gamma^{(n)} = (1, \frac{1}{2^n}, \frac{1}{3^n}, \ldots, \frac{1}{n^n}, 0, 0, \ldots) \)
and \( \gamma^{(m)} = (1, \frac{1}{2^m}, \frac{1}{3^m}, \ldots, \frac{1}{m^m}, \frac{1}{m+1^m}, \ldots, \frac{1}{n^n}, 0, 0, \ldots) \)
For each fixed positive integer \( n \) with \( n > m \)
\[
d(\gamma^{(n)}, \gamma^{(m)}) = \sup \left| \gamma^{(n)} - \gamma^{(m)} \right| = \left\{ 0, 0, \ldots, \frac{1}{m+1}, \ldots, \frac{1}{n}, 0, 0, \ldots \right\}
\]
\[
= \frac{1}{m+1} \to 0 \text{ as } n, m \to \infty
\]
Hence \( \gamma^{(n)} \) is a Cauchy sequence in \( \Gamma_\pi \)
Also \( d(\gamma^{(n)}, 0) = 1 \) so that \( \gamma^{(n)} \in U \), the closed unit ball in \( \Gamma_\pi \).
Note that \( \lim_{n \to \infty} \gamma^{(n)} = \gamma = \left(1, \frac{1}{2^2}, \frac{1}{3^3}, \ldots\right) \) and that \( \gamma \in U \)
Hence each bounded closed set is complete in \( \Gamma_\pi \).
Hence \( \Gamma_\pi \) is quasi complete.
Theorem 4  \( \Gamma_{\pi} \) is not a barrelled space.

Proof:

Let \( A = \left\{ x \in \Gamma_{\pi} : \left| \frac{x}{\pi_n} \right|^n \leq \frac{1}{n} \text{ for all } n \right\} \).

Then \( A \) is an absolutely convex, closed, absorbent set in \( \Gamma_{\pi} \). But \( A \) is not a neighborhood of zero.

Hence \( \Gamma_{\pi} \) is not barrelled.

Theorem 5  \( \Gamma_{\pi} \) is not semi-reflexive.

Proof:

Let \( \{ \delta^{(n)} \} \in U \) the unit closed ball in \( \Gamma_{\pi} \).

But no sequence of \( \{ \delta^{(n)} \} \) can converge weakly to any \( y \in \Gamma_{\pi} \).

Hence \( \Gamma_{\pi} \) is not semi-reflexive.

Theorem 6  The \( \beta \) - dual of \( \chi_{\pi} \) is \( \Lambda_{\pi} \).

Proof:

We shall show that \( \chi_{\pi} \subseteq \Gamma_{\pi} \).

Let \( x \in (x_k) \in \chi_{\pi} \Rightarrow \left\{ \frac{x_k}{\pi_k} \right\} \in \chi \) But \( \chi \subseteq \Gamma \)

Hence \( \left\{ \frac{x_k}{\pi_k} \right\} \in \Gamma \Rightarrow x = \{ x_k \} \in \Gamma_{\pi} \)

Since \( x \) is arbitrary in \( \chi_{\pi} \), we have \( \chi_{\pi} \subseteq \Gamma_{\pi} \)

\( \Gamma_{\pi}^\beta \subseteq \chi_{\pi}^\beta \) But \( \Gamma_{\pi}^\beta = \Lambda_{\pi} \)

Hence \( \Lambda_{\pi} \subseteq \chi_{\pi}^\beta \) . . . . (5)

Next we show that \( \chi_{\pi}^\beta \subseteq \Lambda_{\pi} \)

Let \( \frac{y_k}{\pi_k} \in \chi_{\pi}^\beta \)

\( f(x) = \sum_{k=1}^{\infty} \frac{x_k y_k}{\pi_k \pi_k} \) with \( x \in \chi_{\pi} \)

Take \( x = s^{(n)} \in \chi_{\pi} \)

where \( s^{(n)} = (0, 0, \ldots, \frac{\pi_n}{n!}, 0, \ldots) \)

\( \left\{ (n! \frac{x_n}{\pi_n})^{1/n} \right\} = \{ 0, 0, \ldots, 1, 0, \ldots \} \). Hence converges to zero.

Therefore \( s^{(n)} \in \chi_{\pi} \). Hence \( d(s^{(n)}, 0) = 1 \).
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But \( \left| \frac{y_n}{\pi_n} \right| \leq \| f \| \quad d( s^{(n)}, 0 ) \leq \| f \| \quad \ldots \quad (6) \)

Thus \((y_n)\) is a bounded sequence and hence an analytic rate sequence. In other words \( y \in \Lambda_\pi \). But \( y \) is arbitrary in \( \Lambda_\pi^\beta \).

Therefore \( \Lambda_\pi^\beta \subset \Lambda_\pi \) \ldots \ldots (7)

From (5) and (7) we get

\[ \Lambda_\pi^\beta = \Lambda_\pi \]

**Theorem.** \( \chi_\pi(\Delta) \) is a complete metric space under the metric

\[ d(x,y) = \sup_{(k)} \left\{ \left( k! \left| \frac{\Delta x_k^{(n)}}{\pi_k} - \frac{\Delta y_k}{\pi_k} \right| \right)^{1/k} : k = 1, 2, 3, \ldots \right\} \]

where \( x = (x_k) \in \chi_\pi(\Delta) \) and \( y = (y_k) \in \chi_\pi(\Delta) \).

**Proof:**

Let \( \{x^{(n)}\} \) be a Cauchy sequence in \( \chi_\pi(\Delta) \).

Then given any \( \varepsilon > 0 \) there exists a positive integer \( N \) depending on \( \varepsilon \) such that

\[ d( x^{(n)}, x^{(m)} ) < \varepsilon, \text{ for all } n \geq N \text{ and for all } m \geq N. \]

Hence

\[ \sup_{(k)} \left( k! \left| \frac{\Delta x_k^{(n)}}{\pi_k} - \frac{\Delta x_k^{(m)}}{\pi_k} \right| \right)^{1/k} < \varepsilon \text{ for all } n \geq N \text{ and for all } m \geq N \]

Consequently \( \{k! \frac{\Delta x_k^{(n)}}{\pi_k}\} \) is a Cauchy sequence in the metric space \( C \) of complex numbers. But \( C \) is complete, so \( k! \frac{\Delta x_k^{(n)}}{\pi_k} \rightarrow k! \frac{\Delta x_k}{\pi_k} \) as \( n \rightarrow \infty \).

Hence there exists a positive integer \( n_0 \) such that

\[ \left( k! \left| \frac{\Delta x_k^{(n)}}{\pi_k} - \frac{\Delta x_k}{\pi_k} \right| \right)^{1/k} < \varepsilon \text{ for all } n \geq n_0 \]

In particular, we have

\[ \left( k! \left| \frac{\Delta x_k^{(n_0)}}{\pi_k} - \frac{\Delta x_k}{\pi_k} \right| \right)^{1/k} < \varepsilon \]
Now

\[
\left( k! \left| \frac{\Delta x_k}{\pi_k} \right| \right)^{1/k} \leq \left( k! \left| \frac{\Delta x_k}{\pi_k} - \frac{\Delta x_k(n_0)}{\pi_k} \right| \right)^{1/k} + \left( k! \left| \frac{\Delta x_k(n_0)}{\pi_k} \right| \right)^{1/k} < \varepsilon + 0 \text{ as } k \to \infty
\]

Thus

\[
\left( k! \left| \frac{\Delta x_k}{\pi_k} \right| \right)^{1/k} < \varepsilon \text{ as } k \to \infty
\]

That is \( (x_k) \in \chi_a(\Delta) \)

Therefore \( \chi_a(\Delta) \) is a complete metric space.

REFERENCES


