Asymptotic Behavior for Semi-Linear
Wave Equation with Weak Damping

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Abstract
In this work we study the asymptotic behavior as $t \to \infty$ of the solutions for the initial boundary value problem associated to the semi-linear wave equation with weak damping.

Keywords: Semi-linear wave, weak damping, exponential decay, Lemma of Nakao

1 Introduction
We shall consider the initial boundary value problem associated to the damped semi-linear wave equation,

$$u'' - \Delta u + F(u) + \alpha u' = 0,$$

in the cylinder $Q = \Omega \times (0, T)$, $0 \leq T < \infty$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $\alpha$ is a real positive constant, $F$ is a continuous real function with $sF(s) \geq G(s) \geq 0$, for all $s \in \mathbb{R}^n$ and $G$ is a primitive of $F$. The asymptotic behavior as $t \to \infty$ of the classical solution of equation with $F(u) = u^3$ was studied by Sattinger [7], for small initial data, and Nakao [5] without any restrictions concerning the smallness of the initial data. The exponential decay in the energy norm of weak solutions was proved by Strauss [8] and Nakao [6].
When $F(u) = |u|^\rho u$, $\rho \geq 1$, the existence of global weak solutions was studied by Lions [3] and the asymptotic behavior was proved by Avrin [2]. Existence of global weak solutions when $\alpha = 0$ was proved by Strauss [9]. He also proved that the local energy decay for zero as $t \to \infty$. In this work we prove that the global weak solutions of equation (1) decay exponentially. For our goal we use the following result

**Lemma 1.1** Let $E(t)$ be a nonnegative function on $[0, \infty)$ satisfying

$$\sup_{s \in [t, t+1]} E(s) \leq C_0 (E(t) - E(t+1))$$

where $C_0$ is a positive constant. Then we have

$$E(t) \leq Ce^{-wt} \quad \text{with} \quad w = \frac{1}{C_0 + 1}.$$  

**Proof. 1.1** See page 748 of [6].

### 2 Asymptotic behavior

We use the standard Lebesgue space and Sobolev space with their usual properties as in [1] and in this sense $|\cdot|$ and $\|\cdot\|$ denote the norm in $L^2$ and $H^1_0$ respectively.

**Lemma 2.1** For $F$ Lipschitz, derivable except on a finite number of points with $sF(s) \geq G(s) \geq 0$, for all $s \in \mathbb{R}$, the solution of the initial boundary value problem associated to the equation (1) with initial data $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$ and $G(u_0) \in L^1(\Omega)$ satisfies

$$|u'(t)|^2 + \|u(t)\|^2 \leq C e^{-wt},$$

for all $t \geq 1$, where $C$ and $w$ are positive constants.

**Proof. 2.1** If $F$ is Lipschitz and derivable except on a finite number of points, then there exists a unique solution $u$ of (1), see [9] in the class

$$u \in L^\infty(0, T; H^1_0(\Omega)), \quad (2)$$

$$u' \in L^\infty(0, T; L^2(\Omega)), \quad (3)$$

satisfying

$$u'' - \Delta u + F(u) + \alpha u' = 0 \text{ in } L^2(0, T; L^2(\Omega)) = L^2(Q). \quad (4)$$

Taking the inner product in $L^2(\Omega)$ of (4) with $u'$ we obtain

$$\frac{d}{dt} E(t) + 2\alpha |u'(t)|^2 = 0,$$  

(5)
where
\[ E(t) = |u'(t)|^2 + ||u(t)||^2 + 2 \int_\Omega G(u(t)) \, dx \]
with 
\[ G(s) = \int_0^s F(\xi) \, d\xi. \]

Integrating (5) from \( t \) to \( t+1 \), we get
\[ \int_t^{t+1} |u'(s)|^2 \, ds = \frac{1}{2\alpha}[E(t) - E(t+1)] \equiv D^2(t). \tag{6} \]

The mean value theorem for integrals applied in (6) implies that there exists
\[ t_1 \in \left[ t, t + \frac{1}{4} \right] \quad \text{and} \quad t_2 \in \left[ t + \frac{3}{4}, t + 1 \right] \]
such that \( |u'(t_i)| \leq 2D(t), \ t_i = 1, 2. \)

The inner product in \( L^2(\Omega) \) of (4) with \( u \) implies
\[ \frac{d}{dt}(u'(t), u(t)) - |u'(t)|^2 + ||u(t)||^2 + (F(u(t)), u(t)) + \alpha(u'(t), u(t)) = 0. \]

Integrating \( t_1 \) to \( t_2 \) and applying Cauchy-Schwarz’s inequality and Poincare’s inequality, we get
\[ \int_{t_1}^{t_2} [|u(s)|^2 + (F(u(s)), u(s))] \, ds \leq C_0[|u'(t_1)||u(t_1)| + |u'(t_2)||u(t_2)||]
+ \int_{t_1}^{t_2} |u'(s)|^2 \, ds + \alpha \int_{t_1}^{t_2} |u'(s)||u(s)|| \, ds \]
where \( C_0 \) denote Poincare’s constant in \( \Omega \). Using \( sF(s) \geq G(s) \geq 0, \ |u'(t_i)| \leq 2D(t), \ t_i = 1, 2, \) and (6) we have
\[ \int_{t_1}^{t_2} [|u(s)|^2 + (F(u(s)), u(s))] \, ds \leq 8C_0D(t)\sqrt{E(t)} + (2 + \alpha^2C_0^2)D(t)^2 \equiv H^2(t), \]

from where follows
\[ \int_{t_1}^{t_2} E(s) \, ds \leq D^2(t) + H^2(t), \]

and by mean value theorem for integrals, there exists \( t^* \in [t_1, t_2] \) such that
\[ E(t^*) \leq 2D^2(t) + H^2(t). \tag{7} \]

Integrating (5) from \( t \) to \( t^* \), using (6) and (7) follows that
\[ E(t) \leq E(t^*) + 2\alpha + \int_t^{t+1} |u'(s)|^2 \, ds \leq C_1D^2(t) = \frac{C_1}{2\alpha}[E(t) - E(t+1)], \]
where $C_1 = 4 + (32 + \alpha^2)C_0^2 + 2\alpha$. Then

$$\text{ess sup}_{s \in [t, t+1]} E(s) = E(t) \leq \frac{C_1}{2\alpha}[E(t) - E(t + 1)],$$

and finally by Lemma 1.1 follows that

$$E(t) \leq C e^{-wt},$$

for all $t \geq 1$ with $C$ and $w$ positive constants, and by defintion of $E(t)$ we have $|u'(t)|^2 + ||u(t)||^2 \leq C e^{-wt}$.

Now we are in the position to present our principal result.

**Theorem 2.2** If $F$ is continuous with $sF(s) \geq G(s) \geq 0$, for all $s \in \mathbb{R}$ then the solution of the initial boundary value problem associated to the equation (1) with initial data $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $G(u_0) \in L^1(\Omega)$ satisfies

$$|u'(t)|^2 + ||u(t)||^2 \leq C e^{-wt},$$

for all $t \geq 1$, where $C$ and $w$ are positive constants.

**Proof. 2.3** For $F$ continuous there exists a sequence $(F_k)_{k \in \mathbb{N}}$ with each $F_k$ Lipschitz and derivable except on a finite number of points satisfying $sF_k(s) \geq G_k(s) \geq 0$, for all $s \in \mathbb{R}$ where $G_k(s) = \int_0^s F_k(\xi) d\xi$, such that $F_k \rightarrow F$ uniformly on the bounded sets of $\mathbb{R}$. For each $k \in \mathbb{N}$ let $u_k$ be the solution of (1) when we replace $F$ by $F_k$. For $u$ solution of (1), $u_k$ satisfies (see [9])

$$u_k \rightarrow u \; \text{weakly-star in} \; L^\infty(0, T; H^1_0(\Omega)), \quad (8)$$

$$u_k' \rightarrow u' \; \text{weakly-star in} \; L^\infty(0, T; L^2(\Omega)), \quad (9)$$

$$F_k(u_k) \rightarrow F(u) \; \text{weakly in} \; L^1(\Omega), \quad (10)$$

$$u''_k - \Delta u_k + F_k(u_k) + \alpha u'_k = 0 \; \text{in} \; L^2(Q). \quad (11)$$

From Lemma 2.1 we get $|u'_k(t)|^2 + ||u_k(t)||^2 \leq C e^{-wt}$, for all $t \geq 1$. Let be $t_0 \in [1, T]$, then

$$|u'_k(t_0)|^2 + ||u_k(t_0)||^2 \leq C e^{-wt_0}, \quad (12)$$

$$u_k(t_0) \rightarrow \lambda \; \text{weakly in} \; H^1_0(\Omega) \; \text{as} \; k \rightarrow \infty, \quad (13)$$

$$u'_k(t_0) \rightarrow \eta \; \text{weakly in} \; L^2(\Omega) \; \text{as} \; k \rightarrow \infty. \quad (14)$$

Using (8), (9) and Arzelá-Ascoli’s theorem we have $\lambda = u(t_0)$. Now, we are going to prove that $\eta = u'(t_0)$. For this we consider $\theta \in [t_0, T]$ defined for $\delta > 0$ by

$$\theta(t) = \begin{cases} 
1 & \text{if } t = t_0, \\
-\frac{1}{\delta}(t - t_0 - \delta) & \text{if } t_0 \leq t \leq t_0 + \delta, \\
0 & \text{if } t \geq t_0 + \delta.
\end{cases}$$
It follows from (11) that 
\[ (u_k'' + (u_k, v) + (F_k(u_k), v) + \alpha(u_k', v) = 0 \text{ for all } v \in L^2(\Omega), \text{ in } L^2(0, T). \]

Now multiplying by \( \theta \) and integrating from \( t_0 \) to \( T \) we get for all \( v \in L^2(\Omega) \)
\[
-(u_k'(t_0), v) + \frac{1}{\delta} \int_{t_0}^{t_0+\delta} (u_k'(t), v) dt + \int_{t_0}^{t_0+\delta} a(u_k(t), v) \theta(t) dt \\
+ \int_{t_0}^{t_0+\delta} (F_k(u_k(t)), v) \theta(t) dt + \alpha \int_{t_0}^{t_0+\delta} (u_k'(t), v) \theta(t) dt = 0,
\]
and taking the limit as \( k \to \infty \) using (10) and (14) we obtain
\[
-(\eta, v) + \frac{1}{\delta} \int_{t_0}^{t_0+\delta} (u'(t), v) dt + \int_{t_0}^{t_0+\delta} a(u(t), v) \theta(t) dt \\
+ \int_{t_0}^{t_0+\delta} (F(u(t)), v) \theta(t) dt + \alpha \int_{t_0}^{t_0+\delta} (u'(t), v) \theta(t) dt = 0.
\]

Now taking again the limit as \( \delta \to \infty \) we get \(- (\eta, v) + (u'(t_0), v) = 0 \) and then \( \eta = u'(t_0) \). Therefore, the \( \lim \inf \) as \( k \to \infty \) in (12) implies
\[
|u'(t_0)|^2 + ||u(t_0)||^2 \leq C e^{-w t_0}.
\] (16)

If we consider \( \theta \in C^0[0, T] \) defined for \( \delta > 0 \) by
\[
\theta(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq T - \delta, \\
\frac{1}{\delta}(t - T + \delta) & \text{if } t - \delta \leq t \leq T, \\
0 & \text{if } t = T,
\end{cases}
\]
and repeating the same process we prove that (16) holds for \( t = T \). Finally,
\[
|u'(t)|^2 + ||u(t)||^2 \leq C e^{-w t_0}
\]
for all \( t \geq 1 \) and the proof of theorem is complete.

References


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